A classification of quantum-topological phases of matter

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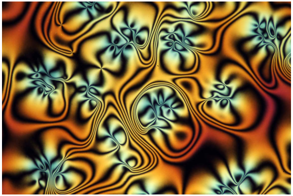
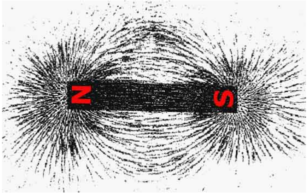
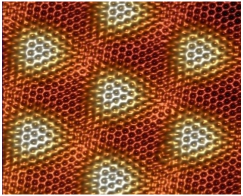
CUHK, 2018/01

Systematic understanding of states of matter

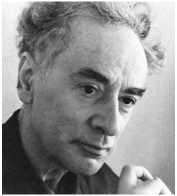






- Rich forms of matter $\leftarrow$ rich types of orders
- A deep insight from Landau: **different orders come from different symmetry breaking.**

  $\rightarrow$ Symmetry breaking theory of orders

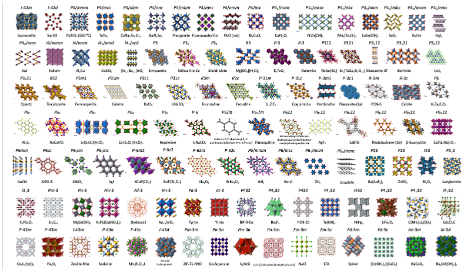
  $\rightarrow$ A corner stone of condensed matter physics

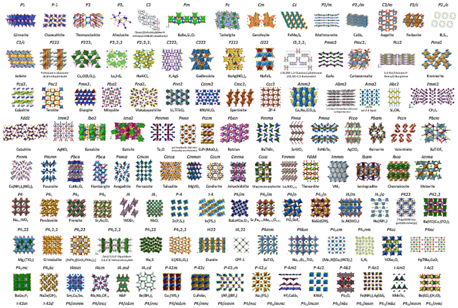


Classification of symmetry breaking phases

- **Symm. breaking phases are classified by a pair** $G_\Psi \subset G_H$
  G_H = symmetry group of the system.
  G_Ψ = symmetry group of the ground states.

- **230 crystals** from group theory



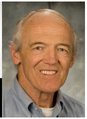


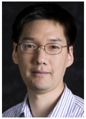
- *For a long time, we thought that Landau symmetry breaking and group theory classify all phases of matter*

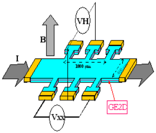
Quantum phases of matter beyond Landau symm. breaking

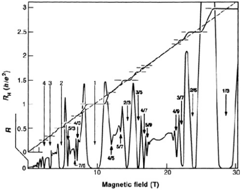
- Beyond symmetry breaking:
  - Quantum Hall states $\Psi_m = \prod (z_i - z_j)^m \mathrm{e}^{-\frac{1}{4l_B^2}\sum |z_i|^2}$, $z_i = x_i + \mathrm{i}\, y_i$
  - Spin liquid states, Organic κ-$(\mathrm{ET})_2\mathrm{X}$ and herbertsmithite

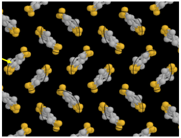











vonKlitzing Dorda Pepper, PRL **45** 494 (1980)

Tsui Stormer Gossard, PRL **48** 1559 (1982)

Kanoda etal, PRL **91** 107001 (2003)

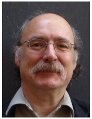
YS Lee etal, PRL **98** 107204 (2007)

- FQH states and spin-liquid states have different phases with no symmetry breaking, no crystal order, no spin order, ...
  $\rightarrow$ **topological order**

Wen, PRB **40** 7387 (1989), IJMP **4** 239 (1990)

Spin-0 chain and spin-1 chain

- Spin-$1/2$ chain with anti-ferromagnetic interaction $H = J\sum_i \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}$ is gapless and almost breaks the $SO(3)$ spin rotation symmetry: $\langle \boldsymbol{S}_i \cdot \boldsymbol{S}_j \rangle \sim |i-j|^{-\alpha_{1/2}}$
  $\to$ Spin-S chain also almost breaks the $SO(3)$ spin rotation symmetry: $\langle \boldsymbol{S}_i \cdot \boldsymbol{S}_j \rangle \sim |i-j|^{-\alpha_S}$, and is gapless.
- Haldane pointed out that integer spin-S chain is gapped
  $\to$ **Haldane phase** Haldane, PRL **50** 1153 (1983)



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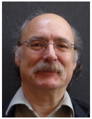

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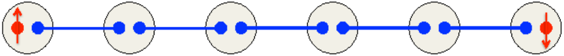
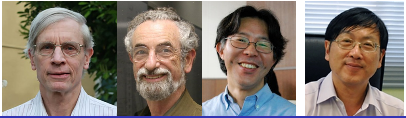
Haldane phases are topological or not topological?

- Spin-0 chain is trivial $\Psi_{\text{grnd}} = |\cdots 00000 \cdots\rangle$.
- AKLT and T.K. Ng pointed out that spin-1 chain is non-trivial due to the emergence of fractionalized spin-$1/2$ at the chain ends.



Affleck-Kennedy-Lieb-Tasaki CMP **115** 477 (88)

Ng PRB **50** 555 (94)

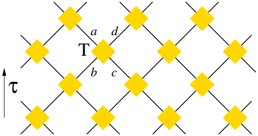


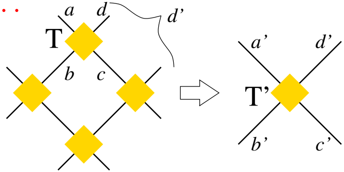
Haldane phases are topological or not topological?

- All 1D short-range correlated states can be written as matrix-product states (MPS)
  Fannes Nachtergaele Werner, CMP **144**, 443 (1992)
  and such matrix-product states are equivalent to product states under wave function RG
  $\rightarrow$ **No topological order in 1D**
  Verstraete Cirac Latorre Rico Wolf, quant-ph/0410227

...

- $\rightarrow$ *Without symmetry breaking, all 1D symmetric states belong to the same trivial product phase (?)*
  $\rightarrow$ *the gapped spin-1 and spin-0 phases are both trivial.*
  Haldane phases are not topological.
- Tensor network entanglement-filtering: Gu Wen, arXiv:0903.1069
  $Z = \sum e^{-S(a,b,\cdots)} = \sum T_{abcd} T_{defg} \cdots$

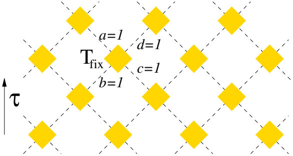




Even symmetric product states can be in different phases

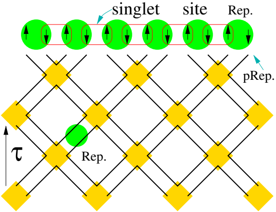
Gu Wen, arXiv:0903.1069

After the entanglement-filtering tensor network renormalization,
(1) the fixed-point tensor T_{fix} may be 1-dimensional (trivial).



(2) the fixed-point tensor T_{fix} may have a corner-double-line structure: $(T_{\text{fix}})_{a_1a_2,b_1b_2,c_1c_2,d_1d_2} = M_{a_2b1}M_{b_2c1}M_{c_2d1}M_{d_2e1}$

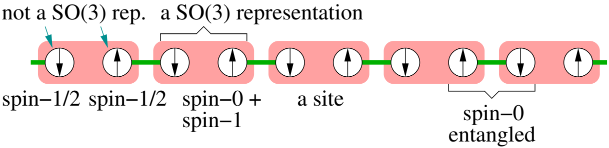
- We find that the corner-double-line structure is stable against any perturbations that respect the symmetry.
- In the presence of a symmetry, even product states without symm. breaking can belong to distinct phases
  → the notion of **SPT order**
  (symmetry protected trivial/topological order).

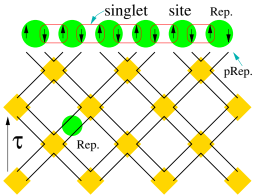


Example of SPT states with $SO(3)$ spin rotation symmetry

After wavefunction RG, the fixed-point $SO(3)$-SPT state has spin-0 $\oplus$ spin-1 per site. Two possible ground states:

- $|\Psi_0\rangle = |\cdots 000000 \cdots\rangle$ (*ie* spin-0 on each site)
- View spin-0 $\oplus$ spin-1 as spin-1/2 $\otimes$ spin-1/2.





ie $|\Psi_0\rangle$ = spin-singlets of the two spin-1/2's on neighboring sites.

- Spin fractionalization spin-0 $\oplus$ spin-1 per site $\to$ spin-1/2 on the boundary $\to$ $SO(3)$-SPT state.

  spin-0 $\oplus$ spin-1 = representation of $SO(3)$.

  spin-1/2 = projective representation of $SO(3)$.

  Fractionalization (*ie* projective representation) on the boundary is the essence of SPT states.

Classification of all 1D gapped quantum phases of matter

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Product states are easy to classify $\rightarrow$

- Projective representation in MPS representation of symmetry and in entanglement spectrum

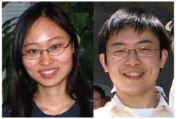
  Pollmann Turner Berg Oshikawa, arXiv:0910.1811

- **All 1D gapped bosonic/fermionic quantum phases are classified by a triple**: $(G_\Psi \subset G_H; \text{pRep}(G_\Psi))$

  $\text{pRep}(G_\Psi)$ is the projective rep. of unbroken symmetry group G_Ψ, describing order beyond symm. breaking.

Chen Gu Wen, arXiv:1008.3745,

Schuch Perez-Garcia Cirac arXiv:1010.3732



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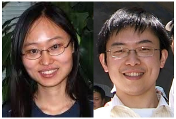

- **Examples**:

  Odd-integer-spin Haldane phase has a non-trivial $SO(3)$-SPT order
  Even-integer-spin Haldane phase has no $SO(3)$-SPT order

  Pollmann Berg Turner Oshikawa, arXiv:0909.4059

Why Haldane phases are not topological? What is topological order?

For gapped systems with no symm.:

- According to Landau theory, no symm. to break
  $\rightarrow$ all systems belong to one trivial phase

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For gapped systems with no symm.: Chen-Gu-Wen arXiv:1004.3835


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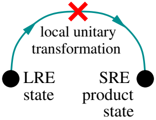
- Thinking about entanglement: there are

- **long range entangled (LRE) states**
- **short range entangled (SRE) states**

$|\text{SRE}\rangle = \;|\text{product state}\rangle \neq |\text{LRE}\rangle$

$|\text{product state}\rangle = |\cdots\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\rangle = \bigotimes_i |\uparrow\rangle_i$



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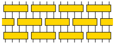
- According to Landau theory, no symm. to break
  $\to$ all systems belong to one trivial phase
- Thinking about entanglement: there are

- **long range entangled (LRE) states** $\to$ many phases
- **short range entangled (SRE) states** $\to$ one phase

$|\text{SRE}\rangle =$  $|\text{product state}\rangle \neq |\text{LRE}\rangle$

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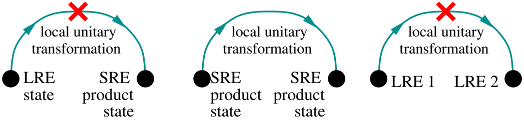

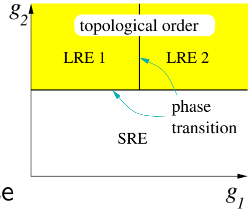

- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases: different patterns of long-range entanglements defined by LU trans.

What is the SPT order in Haldane phases of odd spins?

For gapped systems with a symmetry $H = U_g H U_g^\dagger, \ g \in G$

- **LRE symmetric states** $\to$ many different phases
- **SRE symmetric states** $\to$ one phase (no symm. breaking)

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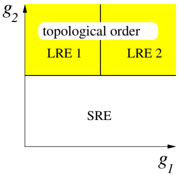
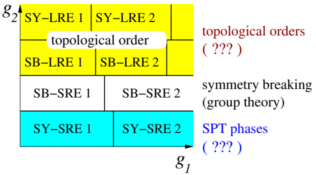
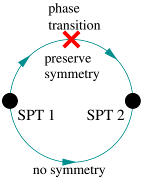
We may call them **symm. protected trivial (SPT)** phase







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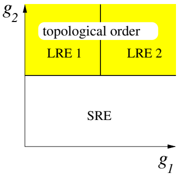
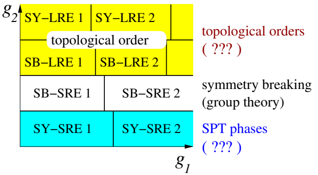
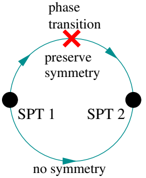
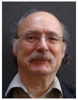
We may call them **symm. protected trivial (SPT)** phase
or **symm. protected topological (SPT)** phase







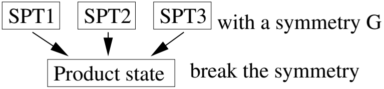
- SPT phases = equivalent class of *symmetric* LU trans.
- Examples: 1D spin-1 gapped phase Haldane 83, 2D TI Kane-Mele 05; Bernevig-Zhang 06, 3D TI Moore-Balents 07; Fu-Kane-Mele 07

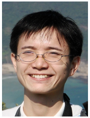
1D  2D  3D 

SPT orders in higher dimensions

- In $0+1$D, the symmetric gapped phases (SPT phases) are classified by the 1-dim. representation of the symmetry group G: 1-dim. $\text{Rep}(G) = \mathcal{H}^{0+1}[G, U(1)]$ *(1st group-cohomology class)*
- In $1+1$D, the symmetric gapped phases (SPT phases) are classified by the projective representation of the symmetry group G: $\text{pRep}(G) = \mathcal{H}^{1+1}[G, U(1)]$.
- **In $d+1$D, the SPT phases are "classified" $\mathcal{H}^{d+1}[G, U(1)]$**

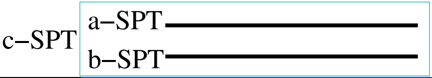
Chen-Gu-Liu-Wen arXiv:1106.4752





Each element in $\mathcal{H}^{d+1}[G, U(1)] \leftrightarrow$ an SPT phase.

- The elements in $\mathcal{H}^{d+1}[G, U(1)] = \{a, b, c, \cdots\}$ form an abelian group: Stacking a-SPT state and b-SPT state give us a c-SPT state: $c = a + b$



Understand the "classification" through topological terms

- Consider an $d+1$D system with symmetry G:
  $S = \int d^d x dt \frac{1}{2\lambda}(\partial g(x^\mu))^2$, symmetry $g(x) \to hg(x), \quad h, g \in G$
  If under RG, $\lambda \to \infty \to$ symmetric ground state described by a fixed point theory $S_{\text{fixed}} = 0$ or $\mathrm{e}^{-S_{\text{fixed}}} = 1$.

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  where $kW = \# k (g^{-1}\mathrm{d}g)^{d+1}$ is a topological term, which is classified by $\{k\} = \mathrm{Hom}(\pi_{d+1}(G), \mathbb{Z})$. $\pi_{d+1}(G)$*: mapping classes.*

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- Fixed point theories (2π-quantized topological terms) $\leftrightarrow$ symmetric phases: *The SPT phases are "classified" by* $Hom(\pi_{d+1}(G),\mathbb{Z})$

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- Another G symmetric system $S=\int d^dxdt\frac{1}{2\lambda}(\partial g(x^\mu))^2+2\pi\mathrm{i}kW$ where $kW=\#k(g^{-1}\mathrm{d}g)^{d+1}$ is a topological term, which is classified by $\{k\}=\text{Hom}(\pi_{d+1}(G),\mathbb{Z})$. $\pi_{d+1}(G)$*: mapping classes.*
  If under RG, $\lambda\to\infty\to$ symmetric ground state described by a fixed point theory $S_{\text{fixed}}=2\pi k\mathrm{i}W$ – *topo. non-linear σ-model*.
- Fixed point theories (2π-quantized topological terms) $\leftrightarrow$ symmetric phases: *The SPT phases are "classified" by Hom$(\pi_{d+1}(G),\mathbb{Z})$*
- But in the $\lambda\to\infty$ limit, $g(x^\mu)$ is not a continuous function. The mapping classes $\pi_{d+1}(G)$ does not make sense.

Understand the "classification" through topological terms

- Consider an $d+1$D system with symmetry G:
  $S = \int d^d x dt \frac{1}{2\lambda}(\partial g(x^\mu))^2$, symmetry $g(x) \to hg(x), \quad h, g \in G$
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- But in the $\lambda \to \infty$ limit, $g(x^\mu)$ is not a continuous function. The mapping classes $\pi_{d+1}(G)$ does not make sense.
- The above idea can be generalized to discrete space-time lattice, $\text{Hom}(\pi_{d+1}(G), \mathbb{Z}) \to \mathcal{H}^{d+1}(G, \mathbb{Z}/\mathbb{R})$, that classifies new topological terms.

What is so spectial about the boundary of SPT states?

- A SPT state has trivial bulk excitations (the same as a tirival symmetric product state).
- But a SPT state has non-trivial boundary described by a "non-tirival" boundary effective theory.

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What do we mean by **"non-tirival" boundary effective theory**?

- The boundary effective theory is WZW-type theory.
- The boundary lattice effective H_{bndry} has **non-on-site** symmetry: $U^{-1}(g) H_{\text{bndry}} U(g) = H_{\text{bndry}},\ g \in G$ with $U(g) \neq \prod_{i \in \text{sites}} U_i(g)$.
- The symmetry G cannot be **gauged** on the boundary. If gauged, the boundary effective theory has G-gauge anomally.

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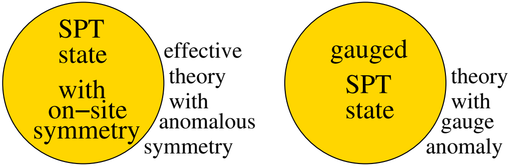
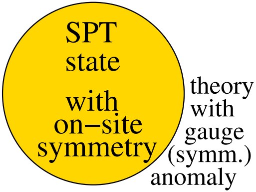
- The symmetry G cannot be **gauged** on the boundary.
  If gauged, the boundary effective theory has G-gauge anomally.

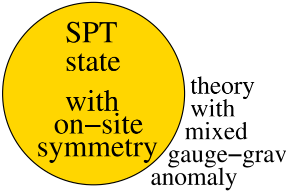
- $d+1$D SPT phases (or group cohomology $\mathcal{H}^{d+1}[G, U(1)]$) classify G-gauge anomalies in one lower dimension (dD space-time).

Wen arXiv:1303.1803



A complete classifiaction of SPT states





- $d + 1$D SPT phases are classified by pure gauge anomalies and mixed gauge-gravty anomalies in one lower dimension:

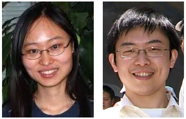
- $d + 1$D SPT phases are classified by $\mathcal{H}^{d+1}[G \times SO_\infty, U(1)]$ (in a many-to-one way), and are realized by $G \times SO_\infty$ non-linear σ-model with topological terms. Wen arXiv:1410.8477

- $d + 1$D SPT phases are classified by coboardism groups.

Kapustin arXiv:1404.6659

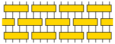
Topological orders = Patterns of long range entanglement

For gapped systems with no symm.: Chen-Gu-Wen arXiv:1004.3835

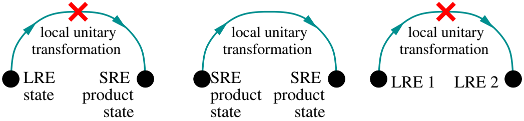


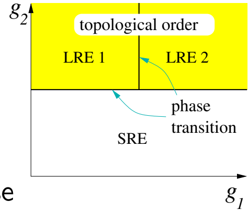
- According to Landau theory, no symm. to break $\rightarrow$ all systems belong to one trivial phase
- Thinking about entanglement: there are

- **long range entangled (LRE) states** $\rightarrow$ many phases
- **short range entangled (SRE) states** $\rightarrow$ one phase

$|\text{SRE}\rangle =$  $|\text{product state}\rangle \neq |\text{LRE}\rangle$

$|\text{product state}\rangle = |\cdots\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\rangle$





- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases: different patterns of long-range entanglements defined by LU trans. = different **topological orders** Wen, Phys. Rev. B40, 7387 (1989)

Some examples of 2+1D topological orders

Abelian topological order: $\to$ fractional statistics

- IQH and Laughlin many-body state Laughlin PRL **50** 1395 (1983)

$$\chi_1 = \prod_{1\leq i<j\leq N}(z_i - z_j)\mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}, \quad \Psi^{F,B}_{\nu=1/n} = \prod(z_i - z_j)^n \mathrm{e}^{-\frac{n}{4}\sum |z_i|^2} = (\chi_1)^n$$

where $z_i = x_i + \mathrm{i}\, y_i$ and $\chi_n = n$ filled Landau levels.

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$$\chi_1 = \prod_{1\le i<j\le N}(z_i - z_j)\,\mathrm{e}^{-\frac14\sum|z_i|^2}, \quad \Psi^{F,B}_{\nu=1/n} = \prod(z_i - z_j)^n\,\mathrm{e}^{-\frac{n}{4}\sum|z_i|^2} = (\chi_1)^n$$

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Non-abelian topological order: $\to$ non-abelian statistics

- $SU(m)_2$ state via slave-particle — Wen PRL **66** 802 (1991)

$$\Psi^B_{SU(2)_2} = (\chi_2)^2,\ \nu = 1; \qquad \Psi^B_{SU(2)_3} = (\chi_3)^2,\ \nu = \frac{3}{2};$$

$\to$ $SU(2)_m$ Chern-Simons effective theory $\to$ non-abelian statistics

- Pfaffien state via correlation CFT — Moore-Read NPB **360** 362 (1991)

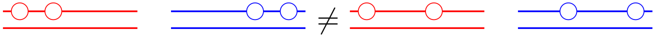
$$\Psi^B_{Pfa} = \mathcal{A}\Big[\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\Big]\prod(z_i - z_j)\,\mathrm{e}^{-\frac14\sum|z_i|^2}, \qquad \nu = 1$$

- *The Pfaffien and $SU(2)_2$ have the same non-abelian statistics*
- *The $SU(2)_3$ state has the Fibonacci non-abelian statistics*

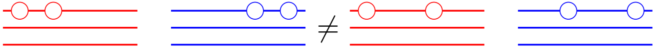
Non-Abelian statistics = fractionalized degrees of freedom

Non-Abelian statistics = Appearance of topological degeneracy even when all the quasiparticles are fully trapped.

- The ground state $(\chi_2)^2 = \chi_2\chi_2$ ($SU(2)_2$ FQH) is non-degenerate.
- Degeneracy D_{deg} of 4 trapped quasiparticles at x_1, x_2, x_3, x_4: many different wave functions: $\chi_2^{x_1 x_2}\chi_2^{x_3 x_4} \neq \chi_2^{x_1 x_3}\chi_2^{x_2 x_4}$



For $(\chi_3)^2$ ($SU(2)_3$ FQH), $\chi_3^{x_1 x_2}\chi_3^{x_3 x_4} \neq \chi_3^{x_1 x_3}\chi_3^{x_2 x_4}$



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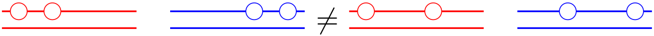

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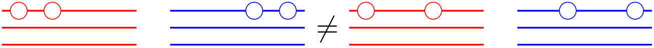


- **Fractionalized degrees of freedom**: N trapped quasiparticle $\to$ degeneracy $D_{\text{deg}}(N)$. Each particle carries degrees of freedom $d = \lim_{N\to\infty}[D_{\text{deg}}(N)]^{\frac{1}{N}}$ (the **quantum dimension** of the particle)

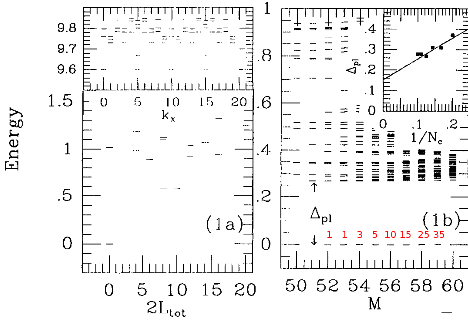
- $d = 2$ for a spin-1/2 particle (a qubit)
  $d = 4$ for a spin-3/2 particle (two qubits)
- For quasiparticle in $\chi_1(\chi_2)^2$ state: $d = \sqrt{2}$ (half qubit).
  For quasiparticle in $(\chi_2)^3$ state: $d = \frac{\sqrt{5}+1}{2}$.

Properties of $\chi_1^k(\chi_m)^n$ $SU(n)_m$ non-Abelain FQH state

.

- The central charge of the edge states:
$$c = mn - \frac{m(n-1)(n+1)}{m+n}$$
- The filling fraction is $\nu = \frac{m}{km+n}$.

- $\chi_1(\chi_2)^2$ $(SU(2)_2)$: $c = \frac{5}{2}$, $\nu = \frac{1}{2}$
- $\chi_1(\chi_3)^2$ $(SU(2)_3)$: $c = \frac{21}{5}$, $\nu = \frac{3}{5}$
- $(\chi_2)^3$ $(SU(3)_2)$: $c = \frac{14}{5}$, $\nu = \frac{2}{3}$



- Pfaffien state:
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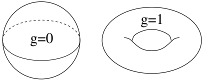
Edge spectrum $\to$ $\mathcal{L}_{edge} = \partial_x\phi(\partial_t\phi - \partial_x\phi) + \mathrm{i}\lambda(\partial_t - \partial_x)\lambda$
$\to$ 1D chiral Majorana fermions on the edge. Wen PRL **70** 355 (93)

Data to measure/define topo.-order/long-range-entangl.

(1) Ψ = space of degenerate ground states: $\Psi \subset \mathcal{V}_{\text{tot}} = \otimes \mathcal{V}_i$

- Topo. degeneracy $D_g \equiv \dim\Psi$, depends on topology of space



Deg.=1 Deg.=D_1

Wen PRB 40, 7387 (89); Wen-Niu PRB 41, 9377 (90)

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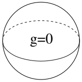
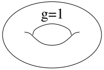


Deg.=1



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(2) **Vector bundle on the moduli space**

i. Consider a torus Σ_1 w/ metrics g_{ij}. **ii**. Different metrics g_{ij} form the moduli space $\mathcal{M} = \{g_{ij}\}$. **iii**. The ground states depend on metrics: $\Psi_\alpha(g_{ij}) \to$ a vector bundle over $\mathcal{M}$ with fiber $\Psi_\alpha(g_{ij})$.

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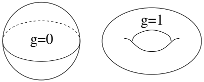

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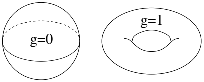

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S, T generate a rep. of modular group: $S^2 = (ST)^3 = C$, $C^2 = 1$

Wen IJMPB 4, 239 (90); KeskiVakkuri-Wen IJMPB 7, 4227 (93)

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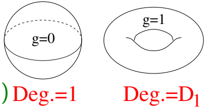

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Conjecture: **The vector bundles from all genus-g Σ_g (*ie* the data (S, T, c), ...) completely characterize the topo. orders**

Classify 2+1D topo. orders (*ie* patterns of entanglement)

via the topological data (S, T, c)

- A 2+1D topological order $\to$ a (S, T, c)
- An arbitary (S, T, c) $\not\to$ a 2+1D topological order

- (S, T, c)'s satisfying **a set of conditions** $\leftrightarrow$ 2+1D topo. orders

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- How to find the conditions? Study topological excitations above the ground states $\to$ unitary modular tensor category theory (UMTC)

Data to describe quasiparticles *ie* non-Abelain statistics

We want to classify 2+1D topological orders by classifying properties of quasiparticles (ie their non-Abelain statistics)

- N: **the number of types** of topological excitations.

- $i = 1$ labels the trivial type, the quasiparticle that can be created by local operators.

- $i = 2, \cdots, N$ label the topological type, the quasiparticles that cannot be created by local operators, such vortices.

Example: The symmetry type in $SO(3)$ symmetric state:

- Spin-0 labels the trivial type, the quasiparticle that can be created by local symmetric operators.

- Spin-$1, 2, \cdots$ label the "topological" type, the quasiparticles that cannot be created by local symmetric operators.

- **Quantum dimension** d_i of type-i quasiparticle = internal degrees of freedom of the type-i quasiparticle.

Example: Spin-S quasiparticle has $d_S = 2S + 1$.

Data to describe quasiparticles *ie* non-Abelain statistics

Fusion of quasiparticles:

Example: Spin-1 $\otimes$ Spin-1 $=$ Spin-0 $\oplus$ Spin-1 $\oplus$ Spin-2, or $1 \otimes 1 = 0 \oplus 1 \oplus 2$. In general $S_1 \otimes S_2 = \bigoplus N^{S_1 S_2}_{S_3} S_3$, with $N^{S_1 S_2}_{S_3} = 1$ for $|S_1 - S_2| \leq S_3 \leq S_1 + S_2$, $N^{S_1 S_2}_{S_3} = 0$ otherwise.

- For generic non-Abelian particles labled by $i = 1, \cdots, N$, their fusions are described by

$$i \otimes j = \bigoplus_{k=1}^{N} N^{ij}_k k.$$

- Fusion identity: $1 \otimes j = j$ where $i = 1$ corresponds to non-topological excitation.
- Anti-particle: $i \otimes \bar{i} = 1$ or $N^{i\bar{i}}_1 = 1$ and $N^{i\bar{i}}_j = 0$ if $j \neq 1$.
- Associativity of fusion: $(i \otimes j) \otimes k = i \otimes (j \otimes k) \to$

$$\sum_m N^{ij}_m N^{mk}_l = \sum_n N^{in}_l N^{jk}_n$$

Fusion coefficients N^{ij}_k and quantum dimension d_i

Let $\boldsymbol{N}_i$ is a matrix with matrix elements $(\boldsymbol{N}_i)_{kj} = N^{ij}_k$, then d_i is the largest eigenvalue of $\boldsymbol{N}_i$.

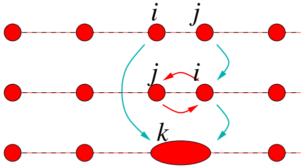
- The number of **topological degenerate ground states** with two fixed particles i and j on a sphere = the number of ways to fuse into 1: $D(i,j) = N^{ij}_1$: $D(i,\bar{i}) = 1$ and $D(i,j) = 0$ is $j \neq \bar{i}$.

- The number of topological degenerate ground states with four fixed particles i, j, k, l on a sphere = the number of ways to fuse into 1: $D(i,j,k,l) = \sum_r N^{ij}_r N^{kl}_{\bar{r}} N^{r\bar{r}}_1 = \sum_r N^{ij}_r N^{kl}_{\bar{r}}$.

- The number of ways to fuse n type-i particles into 1 = $\sum N^{im_{n-2}}_1 \cdots N^{im_1}_{m_2} N^{ii}_{m_1} = (\boldsymbol{N}_i)^{n-2}_{1i} \sim d_i^n$
  $\to$ The quantum dimension d_i is the largest eigenvalue of $\boldsymbol{N}_i$.

Braiding of the quasiparticle

- Particles can also braid $\to$ unitary braided fusion category
- Braiding requires that
  $N_k^{ij} = N_k^{ji}$.



Braiding of the quasiparticle

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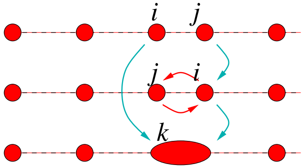
- Braiding requires that $N_k^{ij} = N_k^{ji}$.



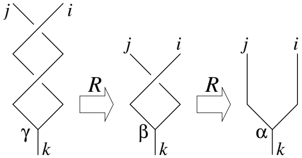
- Braiding $\to$ **mutual statistics** $e^{i\theta_{ij}^{(k)}}$ and non-trivial **spin** s_i

  2π rotation of (i, j) = 2π rotation of k

  2π rotation of (i, j) = 2π rotation of i and j and exchange i, j twice

  $e^{i2\pi s_i} e^{i2\pi s_j} e^{i\theta_{ij}^{(k)}} = e^{i2\pi s_k}$

  or $\theta_{ij}^{(k)} = 2\pi(s_k - s_i - s_j)$



A unitary braided fusion category (UBFC) is a set of quasiparticles with fusion and braiding, which is described by data (N_k^{ij}, s_i)

Relation between (S, T, c) and (N^{ij}_k, s_i, c)

- From (S, T, c) to (N^{ij}_k, s_i, c): Verlinde formula

$$N^{ij}_k = \sum_l \frac{S_{li} S_{lj} (S_{lk})^*}{S_{1l}}, \qquad \mathrm{e}^{\mathrm{i}\, 2\pi s_i}\, \mathrm{e}^{-\mathrm{i}\, 2\pi \frac{c}{24}} = T_{ii}.$$

- From (N^{ij}_k, s_i, c) to (S, T, c):

$$S_{ij} = \frac{1}{\sqrt{\sum_i d_i^2}} \sum_k N^{ij}_k \mathrm{e}^{2\pi \mathrm{i}\, (s_i + s_j - s_k)} d_k, \qquad T_{ii}. = \mathrm{e}^{\mathrm{i}\, 2\pi s_i}\, \mathrm{e}^{-\mathrm{i}\, 2\pi \frac{c}{24}}$$

Conditions on $(N^{ij}_k, s_i, c) \leftrightarrow$ **Conditions on** (S, T, c)
$\rightarrow$ **A theory of unitary modular tensor category (UMTC)**

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Conditions on $(N^{ij}_k, s_i, c) \leftrightarrow$ **Conditions on** (S, T, c)
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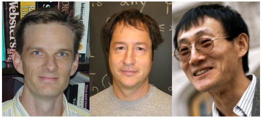
simplified theory of UMTC Rowell-Stong-Wang arXiv:0712.1377



- The standard point of view:
UMTC's are fully characterized by $(N^{ij}_k, F^{ijm;\alpha\beta}_{kln;\gamma\lambda}, R^{ij;\alpha}_{k;\beta})$ (but not one-to-one). Conditions on those data + the equivalent relations $\rightarrow$ a theory of UMTC. *hard to work with*

A simplified theory of UMTC based on (N^{ij}_k, s_i, c)

- **Fusion ring**: N^{ij}_k are non-negative integers that satisfy

$$N^{ij}_k = N^{ji}_k, \quad N^{1i}_j = \delta_{ij},$$
$$\sum_{m=1}^{N} N^{ij}_m N^{mk}_l = \sum_{m=1}^{N} N^{im}_l N^{jk}_m \text{ or } \boldsymbol{N}_k \boldsymbol{N}_i = \boldsymbol{N}_i \boldsymbol{N}_k$$

where $i, j, \cdots = 1, 2, \cdots, N$, and the matrix $\boldsymbol{N}_i$ is given by $(\boldsymbol{N}_i)_{kj} = N^{ij}_k$. We refer N as the rank.

- **Charge conjugation**: N^{ij}_1 defines a charge conjugation $i \to \bar{i}$:

$$N^{ij}_1 = \delta_{\bar{i}j}, \quad \text{such that } \bar{\bar{i}} = i.$$

A simplified theory of UMTC based on (N^{ij}_k, s_i, c)

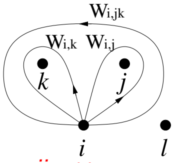
Anderson-Moore 1988; Vafa PLB 206, 421 (88)

- **Rationality theorem**: N^{ij}_k and s_i satisfy

$$\det(W_{i,j})\det(W_{i,k}) = \det(W_{i,jk}) \quad \to \quad \sum_r V^r_{ijkl} s_r = 0 \text{ mod } 1$$

$$V^r_{ijkl} = N^{ij}_r N^{kl}_{\bar r} + N^{il}_r N^{jk}_{\bar r} + N^{ik}_r N^{jl}_{\bar r} - (\delta_{ir} + \delta_{jr} + \delta_{kr} + \delta_{lr}) \sum_m N^{ij}_m N^{kl}_{\bar m}$$

Compute $\det(W_{i,j})$



- Dimension of the fusion space of i, j, k, l:
  $D = \sum_r N^{ij}_r N^{kl}_{\bar r}$
  Thus $W_{i,j}$ is a D dimension unitary matrix.
- $W_{i,j}$ is diagonal in the basis from the fusion $D = \sum_m N^{ij}_r N^{kl}_{\bar r}$.
- The eigenvalue $\mathrm{e}^{\mathrm{i}\theta^{(r)}_{ij}}$ appear $N^{ij}_r N^{kl}_{\bar r}$ times.

$$\det(W_{i,j}) = \prod_r \left(\mathrm{e}^{\mathrm{i}\theta^{(r)}_{ij}}\right)^{N^{ij}_r N^{kl}_{\bar r}} = \prod_r \mathrm{e}^{\mathrm{i}2\pi(s_r - s_i - s_j) N^{ij}_r N^{kl}_{\bar r}}$$

A simplified theory of UMTC based on (N^{ij}_k, s_i, c)

From $(N^{ij}_k, s_i, c) \to (S, T)$

- Let d_i be the largest eigenvalue of the matrix $\boldsymbol{N}_i$. Let

$$S_{ij} = \frac{1}{D} \sum_k N^{ij}_k \mathrm{e}^{2\pi\,\mathrm{i}\,(s_i+s_j-s_k)} d_k, \quad D^2 = \sum_i d_i^2.$$

Then, S satisfies

$$S_{11} > 0, \quad \sum_k S_{kl} N^{ij}_k = \frac{S_{li} S_{lj}}{S_{1l}}, \quad S = S^\dagger C, \quad C_{ij} \equiv N^{ij}_1.$$

- Let $T_{ij} = \mathrm{e}^{\mathrm{i}\,2\pi s_i} \mathrm{e}^{-\,\mathrm{i}\,2\pi \frac{c}{24}} \delta_{ij}$ then (rep. of modular group $SL(2,\mathbb{Z})$)

$$S^2 = (ST)^3 = C.$$

- Let $\nu_i = \frac{1}{D^2} \sum_{jk} N^{jk}_i d_j d_k \mathrm{e}^{4\pi\,\mathrm{i}\,(s_j-s_k)}$. Then $\nu_i = 0$ if $i \neq \bar{i}$, and $\nu_i = \pm 1$ if $i = \bar{i}$.

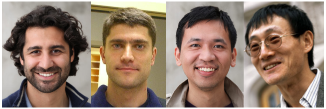
Rowell-Stong-Wang arXiv:0712.1377

2+1D bosonic topo. orders (up to E_8-states) via (N^{ij}_k, s_i, c)

$\zeta_n^m = \frac{\sin(\pi(m+1)/(n+2))}{\sin(\pi/(n+2))}$ Rowell-Stong-Wang arXiv:0712.1377; Wen arXiv:1506.05768

N_c^B	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	wave func.	N_c^B	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	wave func.
1_1^B	1	0					
2_1^B	$1, 1$	$0, \frac{1}{4}$	$\prod(z_i - z_j)^2$	2_{-1}^B	$1, 1$	$0, -\frac{1}{4}$	$\prod(z_i^* - z_j^*)^2$
$2_{14/5}^B$	$1, \zeta_3^1$	$0, \frac{2}{5}$		$2_{-14/5}^B$	$1, \zeta_3^1$	$0, -\frac{2}{5}$	
3_2^B	$1, 1, 1$	$0, \frac{1}{3}, \frac{1}{3}$	(221) double-layer	3_{-2}^B	$1, 1, 1$	$0, -\frac{1}{3}, -\frac{1}{3}$	
$3_{8/7}^B$	$1, \zeta_5^1, \zeta_5^2$	$0, -\frac{1}{7}, \frac{2}{7}$		$3_{-8/7}^B$	$1, \zeta_5^1, \zeta_5^2$	$0, \frac{1}{7}, -\frac{2}{7}$	
$3_{1/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{1}{16}$		$3_{-1/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{1}{16}$	
$3_{3/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{3}{16}$	Ψ_{Pfaffian}	$3_{-3/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{3}{16}$	
$3_{5/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{5}{16}$	χ_2^2 $SU(2)_2$	$3_{-5/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{5}{16}$	
$3_{7/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{7}{16}$		$3_{-7/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{7}{16}$	
$4_0^{B,a}$	$1, 1, 1, 1$	$0, 0, 0, \frac{1}{2}$	Z_2 gauge	4_4^B	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	
4_1^B	$1, 1, 1, 1$	$0, \frac{1}{8}, \frac{1}{8}, \frac{1}{2}$	$\prod(z_i - z_j)^4$	4_{-1}^B	$1, 1, 1, 1$	$0, -\frac{1}{8}, -\frac{1}{8}, \frac{1}{2}$	
4_2^B	$1, 1, 1, 1$	$0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	(220) double-layer	4_{-2}^B	$1, 1, 1, 1$	$0, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}$	
4_3^B	$1, 1, 1, 1$	$0, \frac{3}{8}, \frac{3}{8}, \frac{1}{2}$		4_{-3}^B	$1, 1, 1, 1$	$0, -\frac{3}{8}, -\frac{3}{8}, \frac{1}{2}$	
$4_0^{B,b}$	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, -\frac{1}{4}$	double semion	$4_{9/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, -\frac{1}{4}, \frac{3}{20}, \frac{2}{5}$	
$4_{-9/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{4}, -\frac{3}{20}, -\frac{2}{5}$		$4_{19/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{4}, -\frac{7}{20}, \frac{2}{5}$	
$4_{-19/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, -\frac{1}{4}, \frac{7}{20}, -\frac{2}{5}$	χ_3^2 $SU(2)_3$	$4_0^{B,c}$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, \frac{2}{5}, -\frac{2}{5}, 0$	Fibonacci^2
$4_{12/5}^B$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, -\frac{2}{5}, -\frac{2}{5}, \frac{1}{5}$		$4_{-12/5}^B$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{1}{5}$	
$4_{10/3}^B$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$	$0, \frac{1}{3}, \frac{2}{9}, -\frac{1}{3}$		$4_{-10/3}^B$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$	$0, -\frac{1}{3}, -\frac{2}{9}, \frac{1}{3}$	
5_0^B	$1, 1, 1, 1, 1$	$0, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}$	(223) DL	5_4^B	$1, 1, 1, 1, 1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}$	
$5_2^{B,a}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, \frac{1}{3}$		$5_2^{B,b}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, \frac{1}{3}$	
$5_{-2}^{B,b}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{3}$		$5_{-2}^{B,a}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{3}$	
$5_{16/11}^B$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, -\frac{2}{11}, \frac{2}{11}, \frac{1}{11}, -\frac{5}{11}$		$5_{-16/11}^B$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, \frac{2}{11}, -\frac{2}{11}, -\frac{1}{11}, \frac{5}{11}$	
$5_{18/7}^B$	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, -\frac{1}{7}, -\frac{1}{7}, \frac{1}{7}, \frac{3}{7}$		$5_{-18/7}^B$	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, \frac{1}{7}, \frac{1}{7}, -\frac{1}{7}, -\frac{3}{7}$	

Classification of 2D gapped bosonic quantum phases



- **2D gapped bosonic quantum phases are classified by** : $(G_\Psi \subset G_H; G_\Psi\text{-crossed } MTC; c)$
  Barkeshli Bonderson Cheng Wang, arXiv:1410.4540
  or by $(G_\Psi \subset G_H; \text{REP}(G_\Psi) \subset BFC \subset MTC; c)$
  Lan Kong Wen, arXiv:1602.05946

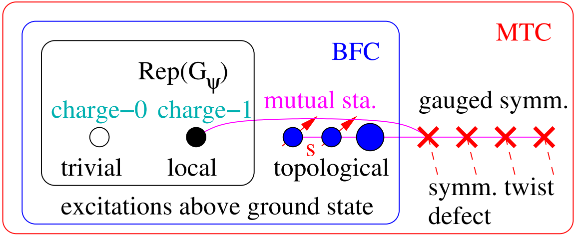
- $G_\Psi \subset G_H \to$ **symmetry breaking**
  $\text{REP}(G_\Psi) \subset BFC \subset MTC; c \to$ **patterns of long-range entanglement** = **topological orders**

- $\text{REP}(G_\Psi) \subset BFC \subset MTC$ are three tensor categories, one contained in another.
- c is the chiral central charge (*ie* the specific heat) of edge states.

But what are **tensor categories**?
- The mathematics to describe patterns of long range entanglement
- The theory to describe gapped excitations/defects.

Classification of 2D gapped bosonic quantum phases

- Carton picture of tensor categories $\mathrm{REP}(G_\Psi) \subset BFC \subset MTC$



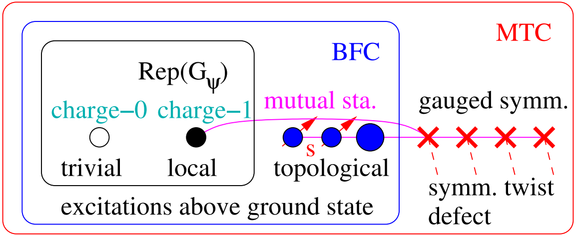
- $\mathrm{REP}(G_\Psi)$ describes trivial and local excitations that can be created by local operators, such as a spin flip. (Not fractionalized)

- Their carry symmetry quantum numbers of the unbroken symmetry $= \mathrm{Rep}(G_\Psi)$
- They carry integer degrees of freedom: **quantum dimension** $d = \dim(\mathrm{Rep}(G_\Psi))$.

Mathematically, $REP(G_\Psi)$ is the symmetric fusion category (SFC) formed by the representations of the unbroken symmetry group G_Ψ.

Classification of 2D gapped bosonic quantum phases

- Carton picture of tensor categories $\mathrm{REP}(G_\Psi) \subset BFC \subset MTC$

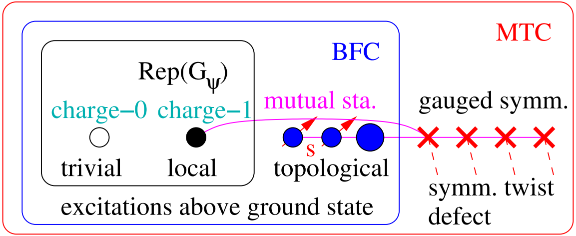


- *BFC* describe all gapped excitations, including the local excitations as well as the **topological excitations** that cannot be created by local operators. Topological excitations are fractionalized:

- Their quantum dimensions d can be non integer $\to$ **fractional degrees of freedom**.
- They may carry **fractional quantum numbers**, *ie* projective representation of G_Ψ and something even more general.
- They may carry **fractional spin** s, such as $s = 1/3$.
- They may carry **fractional/non-abelian statistics**.

Classification of 2D gapped bosonic quantum phases

- Carton picture of tensor categories $\text{REP}(G_\Psi) \subset BFC \subset MTC$



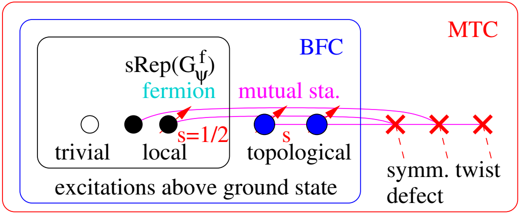
But the *braided-fusion category BFC* does not contain data on how excitations change under symmetry transformation.

- Add symmetry twist defects to expand *BFC* into a *MTC* – *modular tensor category* (non-trivial mutual statistics between all quasiparticles, except the trivial one).

- Braiding with symm. twist defects describes the symm. action.
- 2D topological orders with the same *BFC* but different *MTC* $\rightarrow$ 2D topological orders with the same bulk quasi-particle excitations but different edge states.

Classification of 2D gapped fermionic quantum phases

- **2D gapped fermionic quantum phases are classified by** :
  $(G_\Psi \subset G_H; \mathrm{sREP}(G_\Psi) \subset BFC \subset MTC; c)$



Lan Kong Wen

arXiv:1507.04673

arXiv:1602.05946

- The only change is to allow some rep. of G_Ψ to be fermionic.
- Let $G_\Psi = Z_2^f$ – fermion-number-parity symm. → **classification of all 2D gapped fermionic quantum phases without symmetry** = classification of 2D **fermionic topo. orders**