Appendix:

1. Valuation of a distressed option:

The price $P$ of a distressed option with stochastic liability, which is a function of the firm value $V$ of the underlying asset, the firm value $Q$ of the option writer and the time to maturity $t$ is governed by the partial differential equation

$$
\frac{\partial P(Q,V,t)}{\partial t} = \frac{1}{2} \sigma_Q(t)^2 \frac{\partial^2 P}{\partial Q^2} + \frac{1}{2} \sigma_V(t)^2 V^2 \frac{\partial^2 P}{\partial V^2} + \rho_{QV}(t) \sigma_Q(t) \sigma_V(t) Q \frac{\partial^2 P}{\partial Q \partial V} + [r(t) - d(t)] V \frac{\partial P}{\partial V} + r(t) Q \frac{\partial P}{\partial Q} - r(t) P .
$$

(A.1)

In general, all the model parameters are assumed to be explicitly time-dependent.

To solve this partial differential equation, we first rewrite it in terms of the variables $x_1 = \ln(Q/Q_0)$ and $x_2 = \ln(V/V_0)$, where $Q_0$ and $V_0$ are constants, as follows:

$$
\frac{\partial P(x_1,x_2,t)}{\partial t} = \frac{1}{2} \sigma_Q(t)^2 \frac{\partial^2 P}{\partial x_1^2} + \frac{1}{2} \sigma_V(t)^2 \frac{\partial^2 P}{\partial x_2^2} + \rho_{QV}(t) \sigma_Q(t) \sigma_V(t) \frac{\partial^2 P}{\partial x_1 \partial x_2} + [r(t) - \frac{1}{2} \sigma_Q(t)^2(t)] \frac{\partial P}{\partial x_1} + [r(t) - d(t) - \frac{1}{2} \sigma_V(t)^2(t)] \frac{\partial P}{\partial x_2} - r(t) P .
$$

(A.2)

According to Lo and Hui (2002), the solution of Eq.(A.2) can be formally given by

$$
P(x_1,x_2,t) = \int_{-\infty}^{\infty} dx_1' \int_{-\infty}^{\infty} dx_2' G(x_1,x_2,t;x_1',x_2') P(x_1',x_2',0) ,
$$

(A.3)

where

$$
G(x_1,x_2,t;x_1',x_2') = \frac{\exp \left[ - \int_0^t dt' r(t') \right]}{\sqrt{4\pi c_1(t)} \cdot \sqrt{4\pi c_2(t)} \cdot \sqrt{1 - \eta^2(t)}} \cdot \exp \left\{ \frac{\eta^2(t) z_1(t) z_2(t)}{c_3(t) [1 - \eta^2(t)]} \right\} \cdot \exp \left\{ - \frac{z_1^2(t)}{4c_1(t) [1 - \eta^2(t)]} \right\} \cdot \exp \left\{ - \frac{z_2^2(t)}{4c_2(t) [1 - \eta^2(t)]} \right\}
$$

(A.4)

is the kernel of the pricing equation in Eq.(A.2), and

$$
\eta(t) = \frac{c_3(t)}{2 \sqrt{c_1(t) c_2(t)}} , \quad z_1(t) = x_1' - x_1 - c_4(t) \quad z_2(t) = x_2' - x_2 - c_5(t) , \quad c_1(t) = \int_0^t dt' a_1(t') \quad c_2(t) = \int_0^t dt' a_2(t') \quad c_3(t) = \int_0^t dt' a_3(t') ,
$$

$$
c_4(t) = \int_0^t dt' \left[ r(t') - a_1(t') \right] \quad c_5(t) = \int_0^t dt' \left[ r(t') - d(t') - a_2(t') \right] , \quad a_1(t) = \frac{1}{2} \sigma_Q(t) , \quad a_2(t) = \frac{1}{2} \sigma_V(t) , \quad a_3(t) = \rho_{QV}(t) \sigma_Q(t) \sigma_V(t) .
$$

(A.5)
2. Imposing an early default barrier

For the special case of constant $\sigma_Q$, $\sigma_V$ and $\rho_{QV}$, we can apply the method of images to incorporate an absorbing barrier, i.e. a default barrier, along the $x_1$-axis with a drifted dynamics of the form $X_1(t) = x_{10} - c_4(t) + \beta t$ into our model, where $x_{10} = \ln(Q_0)$ is the pre-defined position of the barrier and the parameter $\beta$ is a real adjustable parameter controlling the movement of the barrier. The corresponding option price is then given by

$$P_{homo}(x_1, x_2, t) = \int_{-\infty}^{\infty} dx_{2}^\prime \int_{0}^{\infty} dx_{1}^\prime K(x_1, x_2, t; x_1^\prime, x_2^\prime) P_{homo}(x_1^\prime, x_2^\prime, 0)$$

$$K(x_1, x_2, t; x_1^\prime, x_2^\prime) = G(x_1, x_2, t; x_1^\prime, x_2^\prime) - G(x_1, x_2, t; -x_1^\prime, x_2^\prime - \frac{a_3}{a_1} x_1^\prime) \exp \left( \frac{\beta}{a_1} x_1^\prime \right), \quad (A.6)$$

where $x_1^\prime = \ln(Q^\prime/Q_0)$ and $x_2^\prime = \ln(V^\prime/V_0)$. It should be noted that this solution vanishes at the barrier; that is, it is the solution associated with the homogeneous boundary condition only. Nevertheless, it is an easy task to extend the solution to satisfy the inhomogeneous boundary condition stated in Eq.(5), by simply adding the trivial solution $\gamma C(x_2, t)$, of the pricing equation in Eq.(A.2). Here $C(x_2, t)$ is the no-default option value and does not depend on $x_1$, and $\gamma$, which lies between 0 and 1, is the proportional recovery made in the event of an early default. Following the same procedure as shown above, the $C(x_2, t)$ can be formally expressed as

$$C(x_2, t) = \int_{-\infty}^{\infty} dx_{2}^\prime G(x_2, t; x_2^\prime) C(x_2^\prime, 0), \quad (A.7)$$

where

$$G(x_2, t; x_2^\prime) = \frac{\exp \left[ - \int_{t}^{0} dt' r(t') \right]}{\sqrt{4\pi c_2(t)}} \exp \left\{ - \frac{z_2^2(t)}{4c_2(t)} \right\}. \quad (A.8)$$

Note that $c_2(t)$ and $z_2(t)$ are the same as those defined above. By imposing the usual final payoff condition of a call option with a strike price $K$:

$$C(x_2, 0) = \begin{cases} 0 & \text{if } V < K \\ V - K & \text{if } V \geq K \end{cases}, \quad (A.9)$$


and performing the integration in Eq.(A.7), we obtain the explicit expression of the no-default option value \( C(x_2, t) \) as follows:

\[
C(x_2, t) = \exp \left[ -\int_0^t dt' r(t') \right] \int_{\ln\left(\frac{K}{V_0}\right)}^\infty dx'_2 \left\{ V_0 \exp(x'_2) - K \right\} \exp \left( -\frac{x'^2_2}{4c_2(t)} \right)
- \exp \left[ -\int_0^t dt' r(t') \right] K \cdot N \left( -\frac{\ln(K/V_0) - x''_2 - 2ta_2}{\sqrt{2ta_2}} \right),
\]

(A.10)

where \( x''_2(t) = x_2 + c_5(t) \) and the \( N(\cdot) \) is the normalized cumulative distribution function. Accordingly, combining \( P_{\text{homo}}(x_1, x_2, t) \) and \( \gamma C(x_2, t) \) yields the desired price function \( P(x_1, x_2, t) \) of the distressed call option, subject to the inhomogeneous boundary condition stated in Eq.(5):

\[
P(x_1, x_2, t) = P_{\text{homo}}(x_1, x_2, t) + \gamma C(x_2, t).
\]

(A.11)

In order to obtain the explicit expression of the distressed option price \( P(x_1, x_2, t) \), we need to impose the final payoff condition stated in Eq.(6), which is equivalent to

\[
P(x_1, x_2, 0) = \begin{cases} 
0 & \text{if } V < K \\
V - K & \text{if } V \geq K
\end{cases}
\]

(A.12)

Such a final payoff condition implies that at \( t = 0 \), \( P_{\text{homo}}(x_1, x_2, t) \) must satisfy the following prescribed condition:

\[
P_{\text{homo}}(x_1, x_2, 0) = \begin{cases} 
0 & \text{if } V < K \\
(1 - \gamma)(V - K) & \text{if } V \geq K
\end{cases}
\]

(A.13)

After performing the integration specified in Eq.(A.6), \( P_{\text{homo}}(x_1, x_2, t) \) is found to be given by:

\[
P_{\text{homo}}(x_1, x_2, t) = \int_0^\infty dx'_1 \int_{\ln(K/V_0)}^\infty dx'_2 (1 - \gamma) V_0 \exp(x'_2) \times
G(x_1, x_2, t; x'_1, x'_2) -
\int_0^\infty dx'_1 \int_{\ln(K/V_0)}^\infty dx'_2 (1 - \gamma) K G(x_1, x_2, t; x'_1, x'_2) -
\]

3
\[
\int_0^\infty dx_1 \int_0^\infty dx_2 (1 - \gamma) V_0 \exp(x_2) \exp\left(\frac{\beta x_1'}{a_1}\right) \times \\
G(x_1, x_2, t; -x_1', x_2' - \frac{a_3}{a_1} x_1') + \\
\int_0^\infty dx_1 \int_0^\infty dx_2 (1 - \gamma) K \exp\left(\frac{\beta x_1'}{a_1}\right) \times \\
G(x_1, x_2, t; -x_1', x_2' - \frac{a_3}{a_1} x_1') \\
\equiv \exp\left[ - \int_0^t dt' r(t') \right] \sum_{n=1}^4 I_n ,
\]

where

\[
I_1 = (1 - \gamma) V_0 \exp\left(x''_2 + a_2 t\right) \cdot N_2 \left( \frac{x''_1 + \theta_1}{\sqrt{2ta_1}}, -\frac{\ln(K/V_0) - x''_2 - \theta_2}{\sqrt{2ta_2}}, \eta \right), \\
I_2 = (\gamma - 1) K \cdot N_2 \left( \frac{x''_1}{\sqrt{2ta_1}}, -\frac{\ln(K/V_0) - x''_2}{\sqrt{2ta_2}}, \eta \right), \\
I_3 = (\gamma - 1) V_0 \exp\left(x''_2 - \frac{\beta}{a_1} x''_1 + \theta_5\right) \cdot N_2 \left( \frac{\theta_3 - x''_1}{\sqrt{2ta_1}}, -\frac{\ln(K/V_0) - x''_2 - \theta_4}{\sqrt{2ta_2}}, \eta \right), \\
I_4 = (1 - \gamma) K \exp\left(-\frac{\beta}{a_1} x''_1 + \theta_8\right) \cdot N_2 \left( \frac{\theta_6 - x''_1}{\sqrt{2ta_1}}, -\frac{\ln(K/V_0) - x''_2 - \theta_7}{\sqrt{2ta_2}}, \eta \right),
\]

\[
\theta_1 = a_3 t, \quad \theta_2 = 2a_2 t, \quad \theta_3 = (a_3 + 2\beta)t, \\
\theta_4 = \frac{2a_1 a_2 t + \beta a_3 t - a_3 x''_1}{a_1}, \quad \theta_5 = \frac{a_1 a_2 t - a_3 x''_1 + \beta^2 t + \beta a_3 t}{a_1}, \\
\theta_6 = 2\beta t, \quad \theta_7 = \frac{a_3(\beta t - x''_1)}{a_1}, \quad \theta_8 = \frac{\beta^2 t}{a_1}, \quad x''_1 = x_1 + c_4(t).
\]

Here \(N_2(\cdot)\) is the normalized bivariate cumulative distribution function. Note that a more explicit form of these results, namely Eqs.(A.11)-(A.15), are presented in Eq.(9) of the main text.