Differential Equations

1. Basic concepts

1.1 Partial differential equations

A differential equation is an equation involving an unknown function and its derivatives.

✧ Ordinary differential equations (ODE): Only one independent variable;
✧ Partial differential equations (PDE): More than one independent variables.

Example: Consider the following are differential equations:

(a) \( \frac{dy}{dx} = 5x + 3 \)

(b) \( 4 \frac{d^3 x}{dt^3} + 2 \left( \frac{dx}{dt} \right)^2 = 1 \)

(c) \( \frac{\partial^2 y}{\partial t^2} - 4 \left( \frac{\partial y}{\partial x} \right)^3 = (x + 1)e^t \)

(d) \( e^x \frac{dy}{dx} + \sin x \frac{d^2 y}{dx^2} = 2y \)

(e) \( \frac{\partial y}{\partial x} = 5x + 3t + 1 \)

Which of them are ODE? ______________________

Which of them are PDE? ______________________

Example:

1. Determines if \( y = x^2 - 1 \) is a solution of the differential equation:

\[ \left( \frac{dy}{dx} \right)^4 - 16y^2 - 32x^2 + 16 = 0 \]

2. Find the solution of the differential equation: \( \frac{\partial^2 y}{\partial t^2} - 4 \left( \frac{\partial y}{\partial x} \right)^3 = (x + 1)e^t \).
1.2 Order of differential equations

The order of a differential equation is the order of the highest derivatives in the equation. For example, the order of an order differential equation: \( \frac{dy}{dx} = 5x + 3 \) is 1; that of \( \frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial y}\right)^3 + 2z^4 + xyz = 2 \) is 2.

Exercises:
(a) \( \frac{d^2 y}{dx^2} = 5x^3 + 3y^3 \); the order is ______________.
(b) \( e^x \frac{d^2 y}{dx^2} + 2 \left( \frac{dy}{dx} \right)^2 = 1 \); the order is ______________.
(c) \( 4 \frac{d^3 y}{dx^3} + \sin x \frac{d^2 y}{dx^2} + 5xy^3 = 0 \); the order is ______________.
(d) \( \frac{\partial^2 y}{\partial x^2} - 4y^2 \frac{\partial^2 y}{\partial t^2} = 0 \); the order is ______________.

2. Separable first-order DE

A first-order differential equation is a **separable equation** if it can be written as
\[
B(y)dy = A(x)dx
\]
(2-1)

Then, the solution of the differential equation can be obtained by simply integrating both sides of (2-1), i.e.,
\[
\int B(y)dy = \int A(x)dx
\]

**Example:** Solve \( \frac{dy}{dx} = \frac{x^2 + 2}{y} \).

Since the differential equation is a separable equation, i.e., \( ydy = (x^2 + 2)dx \). Then, integrate both sides of this equation, we get
\[
\int ydy = \int (x^2 + 2)dx
\]
\[
\frac{1}{2} y^2 = \frac{1}{3} x^3 + 2x + C ,
\]
\[
y^2 = \frac{2}{3} x^3 + 4x + K
\]
where \( C \) and \( K \) are some constants.
Exercise: Solve \( \frac{dy}{dx} = \frac{x^2 y + 2y}{x} \)

3. Exact Equations

Consider the following differential equation:

\[
M(x, y) \, dx + N(x, y) \, dy = 0,
\]

(3-1)

where \( M(x, y) \), \( N(x, y) \), \( M_y(x, y) \) and \( N_x(x, y) \) are continuous functions in both \( x \) and \( y \) in the intervals \( \alpha < x < \beta \), \( \gamma < y < \lambda \). Then, equation (1-1) is exact if and only if

\[
\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x},
\]

(3-2)

for all points in \( \alpha < x < \beta \), \( \gamma < y < \lambda \).

Furthermore, there exists a function \( \psi \) satisfying

\[
\psi_x(x, y) = M(x, y), \quad \psi_y(x, y) = N(x, y),
\]

(3-3)

Equation (2-2) then becomes

\[
\psi_x(x, y) \, dx + \psi_y(x, y) \, dy = 0,
\]

\[
\Rightarrow \quad d\psi(x, y) = 0,
\]

Therefore, the general solution of (2-1) is simply

\[
\psi(x, y) = C.
\]

(3-4)

But how can one find the function \( \psi(x, y) \)?
Example: Solve \((y \cos x + 2xe^y)\,dx + (\sin x + x^2e^y + 2)\,dy = 0\).

\(\checkmark\) Check if it is exact.
Since, \(M(x, y) = y \cos x + 2xe^y\) and \(N(x, y) = \sin x + x^2e^y + 2\), then
\[
M_y(x, y) = \cos x + 2xe^y \\
N_x(x, y) = \cos x + 2xe^y = M_y(x, y)
\]
Therefore, the differential equation is exact.

\(\checkmark\) Next, find the function \(\psi\).
Since
\[
\psi_x(x, y) = M(x, y) = y \cos x + 2xe^y \\
\psi_y(x, y) = N(x, y) = \sin x + x^2e^y + 2
\]
Integrating the first equation gives
\[
\psi(x, y) = y \sin x + x^2e^y + h(y),
\]
where \(h(y)\) are some functions of \(y\) only.

By considering the second equation: \(\psi_y(x, y) = \sin x + x^2e^y + h'(y) = \sin x + x^2e^y + 2\)
It results in \(h'(y) = 2 \Rightarrow h(y) = 2y\), where \(C\) is a constant.

Therefore, the solution to the differential equation is \(y \sin x + x^2e^y + 2y = C\).

Exercise: Solve \(\frac{dy}{dx} = -\frac{2xy}{1 + x^2}\).
4. Method of Integrating Factors

For a linear first-order differential equation has the form:

$$\frac{dy}{dx} + p(x)y = q(x)$$  \hspace{1cm} (4-1)

The solution can be obtained by multiplying an integrating factor \( I(x) = \exp\left( \int p(x)dx \right) \) on the both sides of Eq. (3-1). Since,

\[
I \frac{dy}{dx} + I p y = I q
\]

\[
\frac{d}{dx}(yI) = I q
\]

\[
y I = \int I q dx
\]

Therefore, \( I(x) y(x) = \int I(x) q(x) dx \)

**Example:** Solve \( \frac{dy}{dx} - 3 y = 6 \)

The integrating factor is \( I(x) = \exp\left( \int -3 dx \right) = e^{-3x} \).

Then, \( e^{-3x} y(x) = \int 6e^{-3x} dx = -2e^{-3x} + C \), or \( y(x) = -2 + Ce^{3x} \), where \( C \) is a constant.

**Exercise:** Solve \( \frac{dQ}{dt} + \frac{2}{10 + 2t}Q = 4 \), for \( 0 \leq t < 5 \).
5. Bernoulli equations

A Bernoulli differential equation has the form

\[ \frac{dy}{dx} + p(x)y = q(x)y^n \]  

(5-1)

The solution can be obtained by substituting an auxiliary function \( z = y^{1-n} \) and

\[ \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx} \]. Then, \( y^n \frac{dz}{dx} + p(x)y = q(x)y^n \), or

\[ \frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x) \]  

(5-2)

Eq. (4-2) becomes a linear first order differential equation.

Example: Solve \( \frac{dy}{dx} - \frac{3}{x}y = x^4y^{1/3} \)

Let \( z = y^{2/3} \), so

\[ \frac{dz}{dx} = \frac{2}{3}y^{-1/3}\frac{dy}{dx} \]. The differential equation becomes

\[ \frac{dz}{dx} - \frac{2}{3}z = \frac{2}{3}x^4 \]. The integrating factor is \( I(x) = \exp\left(\int -\frac{2}{3}dx\right) = x^{2} \).

Then, \( x^2z(x) = \int x^2dx = \frac{2}{9}x^3 + C \), or

\( z(x) = \frac{2}{9}x^5 + Cx^2 \), where \( C \) is a constant.

Therefore, \( y(x) = \pm\left(\frac{2}{9}x^5 + Cx^2\right)^{3/2} \).

Exercise: Solve \( \frac{dy}{dx} + xy = xy^2 \)
6. Applications of First-order DE

6.1 Temperature problems

*Newton’s law of cooling* states that the rate of change of temperature of a body is proportional to temperature difference between the body and its surrounding medium, i.e.,

\[
\frac{dT}{dt} = -k(T - T_{\text{env}})
\]

where \( k \) are some positive constants, and \( (T - T_{\text{env}}) \) is the temperature difference between the body and the environment. Although this form of heat loss principle is not very precise, it can provide a good approximation of the change of temperature with time.

**Example:**
A body at a temperature at 3.5 degrees Celsius is placed outdoors where the temperature is 25 degrees Celsius. If after 5 minutes the temperature of the body is 9 degrees Celsius, find (a) how long it will take the body to reach a temperature of 22 degrees Celsius, and (b) the temperature of the body after 20 minutes.

**Solution:**
Since by the Newton’s law of cooling, \( \frac{dT}{dt} = -k(T - T_{\text{env}}), \) then

\[
\frac{dT}{dt} = -k(T - T_{\text{env}})
\]

\[
\int \frac{1}{T - T_{\text{env}}} dT = - \int k dt,
\]

\[
\ln(T - T_{\text{env}}) = -kt + C
\]

\[
T = D \exp(-kt) + T_{\text{env}}
\]

where \( C \) and \( D \) are some constants.

Now when \( t = 0, T = 3.5 \), then \( D = -21.5 \).

When \( t = 5, T = 9 \), then \( k = 0.0591 \). Therefore, \( T = -21.5 \exp(-0.0591t) + 25. \)

(a) When \( T = 22 \), then \( t = 33.32 \) minutes

(b) When \( t = 20 \), then \( T = 18.41 \) degrees Celsius.
6.2 Dilution problems

Suppose a tank of brine solution (塩水) initially contains $V_0$ cm$^3$ and $a$ (g) of salt, so the initial concentration is $a/V_0$ (g/cm$^3$). Now another brine solution of $b$ (g/cm$^3$) is pouring into the tank at a rate $e$ (cm$^3$/min); whereas the stirred solution is leaving at a rate $f$ (cm$^3$/min).

What is the amount of salt in the tank at time $t$?

At time $t$, the volume of the tank is $V = V_0 + et - ft$; the rate of salt adding into the tank is $be$, but the rate of salt leaving out is $fQ/V$, where $Q$ be the amount of salt in the tank at time $t$.

Then, \[
\frac{dQ}{dt} = be - f \frac{Q}{V_0 + et - ft}.
\]

Example:
A tank initially holds 100 (cm$^3$) of brine solution containing 1 (g) of salt. At $t = 0$, another brine solution containing 1 (g/cm$^3$) is poured into the tank at the rate of 3 (cm$^3$/min), while the stirred solution leaves the tank at the same rate. Find (a) the amount of salt in the tank at time $t$ and (b) the time at which the mixture in the tank contains 2 (g) of salt.

Solution:
(a) The amount of salt in the tank at time $t$ is $Q = 100 - 99 \exp(-0.03t)$. (b) When $Q = 2$, $t = 0.338$ (min)
6.3 Electric circuits

When a time varying current, e.g., AC current, passes through a coil, it will induce a back electromotive force (emf) to oppose the changing current. Let $L$ be the inductor of a coil as shown, the back emf is $\varepsilon = -L \frac{dI}{dt}$.

When a potential difference is set up across a capacitor, charges are accumulated on the parallel plates. One gets $Q = CV$, where $C$ is called the capacitance of the capacitor.

Kirchhoff’s loop law states the algebraic sum of the voltage drops in a closed circuit is zero. Consider the circuit as shown, then we have

$$V = IR + \frac{Q}{C} + L \frac{dI}{dt}$$

Exercise:
An RL circuit has an emf of 5 V, a resistance of 50 $\Omega$, an inductance of 1 H, and no initial current. Find the current in the circuit at any time $t$.
Solution: $I = 0.1 - C \exp(-50t)$
7. Applications of Second-order DE

7.1 RCL circuit problems

Example:
An RCL circuit connected in series has \( R = 180 \, \Omega \), \( C = 1/280 \, \text{F} \), \( L = 20 \, \text{H} \), and an applied voltage \( V(t) = 10 \sin t \). Assuming no initial charge on the capacitor, but an initial current of 1 A at \( t = 0 \) when the voltage is first applied, find the subsequent charge on the capacitor.

![RCL Circuit Diagram]

Solution:
By Kirchhoff’s loop law, \( V = IR + \frac{Q}{C} + L \frac{dI}{dt} \), we have

\[
10 \sin t = 180I + 280Q + 20 \frac{dI}{dt}.
\]

Since \( I = \frac{dQ}{dt} \), then the differential equation becomes

\[
10 \sin t = 180I + 280Q + 20 \frac{d^2Q}{dt^2},
\]

or

\[
\frac{d^2Q}{dt^2} + 9 \frac{dQ}{dt} + 14Q = 0.5 \sin t.
\]

This is a second-order differential equation with constant coefficients. I suppose you are able to do the rest.
7.2 Buoyancy problems

A body in liquid experiences a buoyant upward force equal to the weight of the liquid displaced by that body. This is famous Archimedes' principle.

Suppose a cylinder of density $\rho$, radius $r$ and height $H$. At equilibrium, $h$ of the cylinder height is submerged in liquid. The weight of the cylinder is $mg$; whereas the upward force is $\pi r^2 h \rho g$ by the Archimedes’ principle. Since it is at equilibrium, we then get $\pi r^2 h \rho g = mg$.

Now, the cylinder is raised out of the water by $x(t)$ from equilibrium, the upward force is less than its weight $mg$, so it is no longer at equilibrium. The net force acting on the cylinder is

$$F = \frac{d^2 x}{d t^2} = \pi r^2 (h - x) \rho g - mg$$

$$\frac{d^2 x}{d t^2} = -\pi r^2 \rho g x$$

In cases where $r$, $\rho$ and $g$ are constants, this is again a second-order differential equation with constant coefficients.

**Example:** (a) Determine whether a cylinder of radius 4 cm, height 10 cm, and mass 15 g can float in a deep pool of water of density 0.03617 g/cm³. (b) If it is released with 20% of its length above the water line with a velocity of 5 cm/s in the downward direction, write an expression for the motion of the cylinder.

**Solution:**

(a) Yes (b) $x(t) = -0.0045 \sin(11.01 t) + 0.0025 \cos(11.01 t)$. 