Homework I

(Matrices and Determinants)

Due-Date: 04/11/2006 (Sat.)

1. If \( A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \), \( S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \), prove that \( D = S^{-1}AS \) is a diagonal matrix.

Verify that \( \text{Tr}(S^{-1}AS) = \text{Tr}(A) \), and that \( \text{det}(A) = \text{det}(S^{-1}AS) \), (\( \text{Tr} \) denotes trace, the sum of the diagonal elements). Show also that \( D^n = S^{-1}A^nS \), where \( n \) is a positive integer.

[25 marks]

2. (a) Show that the general matrix \( A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \) may be written in the form \( A = B + C \), where \( B \) is a symmetric matrix (i.e. \( B^T = B \)) and \( C \) is a skew-symmetric matrix (i.e. \( C^T = -C \)). Show that \( B^2 \) is a symmetric matrix and that \( C^2 \) is a diagonal matrix.

[12 marks]

(b) Given \( E = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), where \( b \neq 0 \), and if \( E^2 = I \), evaluate \( c \) and \( d \) in terms of \( a \) and \( b \). State briefly why this matrix \( E \) is of interest.

[13 marks]

3. Verify that if \( A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \), then \( A^2 - 4A - 5I = 0 \), and hence evaluate \( A^{-1} \).

[25 marks]

4. (i) \( A, B \) and \( C \) are known matrices of size \( n \times n \). Solve the following matrix equations for the \( n \times n \) square matrix \( X \):

(a) \( B - AX = 3X + C \),

(b) \( (X + 2C)B = 2(A - X)C \)

[8 marks]

[8 marks]

(ii) Prove that if \( D \) and \( E \) are two \( n \times n \) matrices and \( D \) is nonsingular, then

\[ \text{det}(I - D^{-1}ED) = \text{det}(I - E). \]

[9 marks]