Solution to Homework 1 of Module 1 (Part 2)

1. Solve \( \frac{dx}{dx} - x = -x \), subject to the initial condition \( z(0) = -4 \).

   **Answer:** The integrating factor is \( \mu = \exp \left( \int -x \, dx \right) = \exp(-x^2/2) \).

   Multiplying the factor to both sides of the differential equation, we have

   \[
   \exp(-x^2/2) \frac{dz}{dx} - x \exp(-x^2/2)z = -x \exp(-x^2/2)
   \]

   \[
   \frac{d}{dx} \left[ z \exp(-x^2/2) \right] = -x \exp(-x^2/2)
   \]

   \[
   z \exp(-x^2/2) = \int -x \exp(-x^2/2) \, dx + C = 1 + C \exp(x^2/2)
   \]

2. Solve \( \frac{dy}{dx} + y = \sin(x) \).

   **Answer:** The integrating factor is \( \mu = \exp \left( \int \sin(x) \, dx \right) = \exp(x) \).

   Multiplying the factor to both sides of the differential equation, we have

   \[
   \exp(x) \frac{dy}{dx} + y \exp(x) = \sin(x) \exp(x)
   \]

   \[
   \frac{d}{dx} \left[ y \exp(x) \right] = \sin(x) \exp(x)
   \]

   \[
   y \exp(x) = \int \sin(x) \exp(x) \, dx = \frac{1}{2} \exp(x) \sin(x) - \frac{1}{2} \exp(x) \cos(x) + C
   \]

3. Solve \( \frac{dy}{dx} + y - \frac{1}{1 + e^x} = 0 \).

   **Answer:** The integrating factor is \( \mu = \exp \left( \int \frac{1}{1 + e^x} \, dx \right) = \exp(x) \).

   Multiplying this integrating factor to the DE, we have

   \[
   y \exp(x) = \int \frac{1}{1 + e^x} \, dx = \frac{1}{2} \exp(2x) + \frac{1}{2} \exp(x) \sinh(x) + \frac{x}{2} + C
   \]

4. Solve \( \frac{dy}{dx} + 2xy = x \).

   **Answer:** The integrating factor is \( \mu = \exp \left( \int 2y \, dx \right) = \exp(x^2) \).

   Multiplying this integrating factor to the DE, we have

   \[
   \exp(x^2) \frac{dy}{dx} + 2xy \exp(x^2) = x \exp(x^2)
   \]

   \[
   \frac{d}{dx} \left[ x \exp(x^2) \right] = \frac{1}{2} \exp(x^2) + C
   \]

5. Solve \( \cosh(x) \frac{dy}{dx} + \sinh(x) y = \cosh^3 x \), subject to \( y(0) = \frac{1}{4} \).

   **Answer:** We rewrite \( \cosh(x) \frac{dy}{dx} + \sinh(x) y = \cosh^3 x \) as

   \[
   \frac{dy}{dx} + \tanh(x) y = \cosh x
   \]

   The integrating factor is \( \mu = \exp \left( \int \tanh(x) \, dx \right) = \cosh(x) \).

   Multiplying this integrating factor to the DE, we have \( \cosh(x) \frac{dy}{dx} + \sinh(x) y = \cosh^3 x \) (The original one!), and hence

   \[
   \frac{d}{dx} [ y \cosh(x) ] = \cosh^3 x
   \]

   Integrating both sides, we have

   \[
   y \cosh(x) = \int \cosh^3 x \, dx = \int \frac{1}{2} \cosh(2x) + \frac{1}{2} \cosh(x) \, dx = \frac{1}{4} \sinh(2x) + \frac{x}{2} + C
   \]

   \[
   y = \frac{\sinh(2x) + x}{4 \cosh(x) + 2 \cosh(x) \cosh(x)} + C
   \]
6. (a) The solution is:

\[
\frac{dv}{dt} = \frac{ck - mg}{m_0 - kt}
\]

\[
(m_0 - k)\frac{dv}{dt} = ck - (m_0 - kt)g
\]

\[
v(t) = \frac{ck}{m_0 - kt} - g t
\]

\[
\int_{v_0}^{v(t)} dv' = \int_{t_0}^{t} \left( \frac{ck}{m_0 - kt'} - g t' \right) dt'
\]

\[
v(t) = -c \ln\left( \frac{m_0 - kt}{m_0} \right) - gt
\]

(b) When the fuel is exhausted, \(kt = 0.8 m_0\), hence the velocity is

\[
v(t) = \frac{2500 \ln\left( \frac{m_0}{0.2m_0} \right) - 9.8 \times 100}{m/s}
\]

\[
= (2500 \ln5 - 980) \text{ m/s}
\]

\[
= 3043.6 \text{ m/s}
\]
Solution to Homework 2 of Module 1 (Part 2)

2. Solve \( \frac{d^2Q}{dt^2} - \frac{5}{2} \frac{dQ}{dt} + 7Q = 0 \).
   \[ \text{Answer:} \quad Q = A e^{(\sqrt{7}/2)t} + Be^{-(\sqrt{7}/2)t}. \]

3. Solve \( \frac{d^2x}{dt^2} - 10 \frac{dx}{dt} + 25x = 0 \).
   \[ \text{Answer:} \quad x = A e^{t} + Be^{-t}. \]

4. Solve \( \frac{d^2y}{dt^2} - 7y = 0 \).
   \[ \text{Answer:} \quad y = A e^{t/\sqrt{7}} + Be^{-t/\sqrt{7}}. \]

5. Solve \( \frac{d^2u}{dt^2} - 2 \frac{du}{dt} + 4u = 0 \).
   \[ \text{Answer:} \quad u = A e^{2t/\sqrt{2}} + Be^{-2t/\sqrt{2}}. \]

6. Solve \( \frac{d^2y}{dx^2} - \frac{3}{5} \frac{dy}{dx} - 5y = 0 \).
   \[ \text{Answer:} \quad y = A e^{(3+\sqrt{29}/2)x} + Be^{(3-\sqrt{29}/2)x}. \]

7. Answer: \( \frac{dy}{dx} = \frac{4x^3}{y + \cos y} \)
   can be rewritten as \( (y + \cos y)dy = 4x^3dx \). Integrating both sides, we have
   \[ \frac{y^2}{2} + \sin y = \frac{4x^4}{3} + C. \]
   Since \( y(1) = \pi \), we have \( C = \frac{\pi^2}{3} - \frac{\pi^2}{2} \).
   and hence \( \frac{y^2}{2} + \sin y = \frac{4}{3}(x^4 - 1) + \frac{\pi^2}{2}. \)

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