Extinction paradox and actual power scattered in light beam scattering: a two-dimensional study

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The extinction paradox is examined by applying partial-wave analysis to a two-dimensional light beam interacting with a long transverse cylinder without absorption, assuming always short wavelengths. We show that the (conventional) power scattered, $P_{\text{sca}}$, except for a very narrow beam hitting a transparent cylinder on axis, is always double the power directly intercepted by the scatterer, $P_{\text{itc}}$, including a zero result for $P_{\text{itc}}$ when the incident beam is basically off the material surface. This contradicts the interpretation that attributes one half of $P_{\text{sca}}$ to edge diffraction by the scatterer. Furthermore, we identify the shadow-forming wave (SFW) from the partial-wave sum in the forward direction and show that the actual power scattered or, equivalently, the power depleted from the incident beam is essentially to one unit of $P_{\itc}$, for a narrow beam, gets larger for a broader beam, and approaches $2P_{\itc}$ for a very broad beam. The larger value in the latter cases is due to the extent of divergence of the SFW beam out of the incident beam at distances well beyond the Rayleigh range. © 2004 Optical Society of America

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1. INTRODUCTION

The extinction paradox in light scattering, and a similar one in quantum scattering, are well-known in the literature.\textsuperscript{1−13} In the case of a sphere without dissipation, for example, it refers to the theoretical value of the power scattered, $P_{\text{sca}}$ (or, equivalently, the scattering cross section $\sigma_{\text{sca}}$), becoming twice as much as the power directly intercepted by the scatterer, $P_{\text{itc}}$ (or the physical cross section of the scatterer), in the limit of zero wavelength. This result appears to have been first explicitly obtained by Stratton and Houghton\textsuperscript{1} through partial-wave analysis. A prevailing interpretation of this twice-as-much paradox is that one half of $P_{\text{sca}}$ is due to the deflection (including reflection and refraction) of light hitting directly on the object and the other half is due to edge diffraction by the sphere.\textsuperscript{8,11−13} This means that those off-surface light rays at an impact parameter as large as $\sqrt{2}$ times the radius are influenced by the scatterer even in the limit of zero wavelength, in contradiction to the concept of light rays in geometrical optics. Although the edge diffraction is always there, it involves only the light rays passing within a few wavelengths of the physical boundary of the scatterer and thus the resultant extra scattering cross section should be negligibly small in the limit of geometrical optics. Brillouin\textsuperscript{14} was fully aware of this problem, and he proposed a modified scattering cross section that differs from $\sigma_{\text{sca}}$ by excluding the scattered power within a small angle of divergence in the forward direction. Keller\textsuperscript{9} has used the concept of light rays and energy conservation to rule out the role of edge diffraction in the interpretation of the paradox. More recently, Lok\textsuperscript{12} has doubted the interpretation in his study of a narrow beam striking the sphere head on; yet he maintained the edge-diffraction interpretation in the case of a beam broader than the radius of the sphere.

The conventional $P_{\text{sca}}$ (or $\sigma_{\text{sca}}$) is calculated by using only the so-called scattered field, which is not the total field; the calculation of the twice-as-much result in the limit of geometrical optics is perfectly correct. Our aim here is, first, to examine the interpretation of the paradox by using general partial-wave analysis. We consider a simpler situation, namely, the scattering of a two-dimensional (2D) wave by a long transverse (circular) cylinder of radius $a$, the 2D analog of the three-dimensional (3D) scattering by a sphere. For a plane incident wave, it is known in the literature (Ref. 5 and also the references cited in Ref. 3) that the corresponding scattering “cross section” (per unit length of the cylinder) in the limit of zero wavelength is $4\alpha$, twice the physical “cross section” $2\alpha$. Since a light ray in geometrical optics is really a highly collimated light beam and the incident wave is also really beamlike in experiments, we investigate the scattering of a 2D wave beam by the cylinder in which the beam can be either narrow or broad compared with $a$. In particular, we let the beam scan across the cylinder, and, at each position, we calculate the scattering efficiency $\eta$, i.e., the ratio of $P_{\text{sca}}$ to the incident power of the beam. The formulation and the necessary formulas are developed in Section 2 assuming no absorption. Numerical and analytic results of $\eta$ for a dielectric cylinder and for a perfectly conducting cylinder are presented in Section 3 with various size parameters $s (=2\pi a/\lambda = ka$, $\lambda$ is the wavelength) and various half-widths $w$ of the incident beam, satisfying always $s \gg 1$ and $kw \gg 1$. We find that $P_{\text{sca}}$ is almost always approximately twice the power directly intercepted by the cylinder, $P_{\text{itc}}$, including the case.
of a zero \(P_{\text{sc}}\) once the beam is basically off the cylinder. This is in conflict with the interpretation involving edge diffraction. The only exception to the doubled result is for the case of a narrow beam hitting a large transparent cylinder on axis, where the scattering efficiency versus \(s\) is found to oscillate substantially about 2, a result also obtained numerically by Lock\(^{12}\) in his study of a 3D narrow beam striking a transparent sphere head on. We show that such an oscillation also occurs for a beam transmitting through a transparent layer and is thus not really a surprise.

An explanation of the paradox that need not rely on edge diffraction has been given before in studies of both optical and quantum scattering in the case of a plane incident wave.\(^{9,10}\) It is based on the recognition that the actual field outside is the superposition of the scattered field and the incident field. Hence the scattered field naturally consists of a reflection (or deflection) part and another part called the shadow-forming wave (SFW) (in fact, a beam), which cancels the incident wave in the forward direction right behind the scatterer.\(^{3,5,9,10,14}\) The paradox can therefore be understood by realizing that the reflection part and the SFW part, for short wavelengths, each contributes a equal amount that is just equal to the physical cross section of the scatterer. Yet the question becomes whether \(\sigma_{\text{sc}}\), which is calculated not by using the actual field, gives the measurable result. In general scattering theory, it is asserted that “anything other than the incident wave forms part of the scattered wave, and contributes an asymptotically outgoing wave at large distance” (Peierls’s words in Ref. 10, p. 9), implying that \(\sigma_{\text{sc}}\) is indeed the actual scattering cross section. But this cannot be right for a narrow incident beam hitting a large reflecting object (or a hard sphere in quantum scattering). As such, we want to study the paradox in a more general and unified way by considering an incident wave beam that can be either narrow or broad, approaching a ray or a plane wave in the two extremes, using our 2D formulation developed in Section 2 and always bearing in mind that the total field gives the measurable result. In Section 4, we first identify the SFW beam from the partial-wave sum and then show that, for a narrow incident beam of waist smaller than the diameter of the cylinder, the SFW beam cancels the incident beam completely in the forward direction and, as a result, the actual power scattered is simply one unit of \(P_{\text{inc}}\). The paradox does not exist at all. For a very broad incident beam, we demonstrate that the SFW beam diverges completely outside the incident beam in the deep Fraunhofer region or well beyond the Rayleigh range in the language of beam propagation. The actual power scattered is indeed twice the power intercepted.

In optical experiments, the actual scattering cross section is commonly measured in terms of the power depleted from a broad incident beam.\(^{2,15}\) We study this in Section 5, again within our 2D formulation, by allowing the beam width to vary. In view of the possible effect from asymptotically outgoing waves at large distance, we introduce a simple method to calculate the power depleted in the region well beyond the Rayleigh range of the incident wave beam and show that it is basically equal to one unit of \(P_{\text{inc}}\) (corresponding to an actual scattering “cross section” of 2\(a\)) for \(w = a\), increases gradually for a broader beam \((w \gg a)\), and approaches essentially the value of 2\(P_{\text{inc}}\) (corresponding to an actual scattering “cross section” of 4\(a\)) in the limit of \(w \gg a\). A discussion and conclusions are given in Section 6.

2. SCATTERING OF A TWO-DIMENSIONAL LIGHT BEAM BY A CYLINDER

A. General Consideration

Consider a single-frequency 2D light beam propagating in free space along the \(y\) direction and centered (or focused) at a position \(r_0 = (x_0, y_0)\), say. A cylinder is then placed with its axis coincident with the \(z\) axis. The cross section of the (circular) cylinder and the beam are shown schematically in Fig. 1. For simplicity, we assume TE polarization, so that the electric field has only the \(x\) component. Everything is independent of \(z\). As is well-known, the beam electric field can always be expressed as a linear superposition of plane-wave solutions as follows:

\[
E_{\text{inc}}(x, y) = \int_{-k}^{k} dk_x \tilde{E}(k_x; r_0) \exp(ik_x x + ik_y y),
\]

where \(\tilde{E}\), which contains \(r_0\) as a parameter, is any function of \(k_x, k_y = \sqrt{k^2 - k_z^2}\), and the time factor \(\exp(-i\omega t)\) has been omitted. Note that, with the integration limits at \(\pm k\) (i.e., \(\pm \omega/c\)), we have excluded component waves decaying along the \(y\) direction; such excluded waves are believed not present in any wave beam propagating in free space. Note also that \(\tilde{E} = \delta(k_x)\) for a plane incident wave of unit field amplitude.

Writing \(k_x x + k_y y = kr \sin(\phi + \gamma)\), where \((r, \phi)\) is the position in polar coordinates and the angle \(\gamma\) is defined by

\[
\sin \gamma = k_z / k,
\]

we can express the exponential function in Eq. (1) in terms of the Bessel functions of the first kind. Hence we can readily obtain

\[
E_{\text{inc}}(x, y) = \sum_m \beta_m J_m(kr) \exp(i m \phi)
\]

in terms of the partial waves each of a mode number (or an “angular momentum index”) \(m\), where the summation is over all positive and negative integers including 0, and

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**Fig. 1.** Sketch of the cross section of a (circular) cylinder and a 2D incident wave beam of width 2\(w\) focused at \((x_0, 0)\).
\[ \beta_m = \int_{-k}^{k} dk_x \tilde{E}(k_x; r_0) \exp(ik \gamma) \]  

is called the beam-shape coefficient, which is obviously independent of \( m \) in the case of a plane incident wave. The partial waves are independent of one another, and, for a circular cylinder, each partial wave in the incident field is related to the partial wave in the internal and scattered fields of the same mode number through the boundary conditions. The scattered field can subsequently be obtained and expressed in terms of the scattering phase shift \( \delta_m \) (Ref. 16):

\[ E_{\text{sc}}(x, y) = i \sum_m \beta_m(\sin \delta_m)H_m^{(1)}(kr)\exp(im \phi + i \delta_m), \]  

where

\[ \tan \delta_m = \frac{J_m(ns)J'_m(s) - nJ'_m(ns)J_m(s)}{J_m(ns)Y'_m(s) - nY'_m(ns)Y_m(s)} \]  

if the cylinder is a transparent dielectric of (real) index of refraction \( n \) or

\[ \tan \delta_m = J_m(s)/Y_m(s) \]  

if the cylinder is a perfect conductor. Here \( H_m^{(1)} \) \((=J_m + iY_m)\), \( Y_m \) being the Neumann function) is the Hankel function with outgoing wave at infinity and the prime in \( J'_m \) and \( Y'_m \) means the derivative of the function with respect to its argument. Note that \( \delta_m \) is \( s \) dependent while \( \beta_m \) is not. Note also that the well-known plane-wave result is recovered when \( \beta_m \) is a constant for all \( m \).

Since Eq. (5) is of the same form as that in the case of a plane incident wave, the (time-averaged) scattered power (per unit length of the cylinder) for the beam case can be readily obtained:

\[ P_{\text{sca}} = \frac{2}{\mu_0 \omega} \sum_m |\beta_m|^2 \sin^2 \delta_m. \]  

However, we want to stress that this result is obtainable by integrating the (time-averaged) Poynting vector over any closed path around the cylinder; there is no need to have recourse to the properties of the Hankel functions at infinity.

### B. Beam-Shape Coefficient for a "Good" Beam

To carry out any explicit calculation and comparison, we need to know the beam-shape coefficient. The simplicity in the 2D case is that we have the remarkably simple formula

\[ \beta_m = F_{\text{inc}}(m/k, 0) \]  

for a "good" beam, i.e., a highly collimated beam of small angular divergence; it is simply equal to the value of the field at a perpendicular distance of \( m/k \) from the axis of the cylinder. The formula can be easily obtained by noting that \( \tilde{E}(k_x) \) in Eq. (1) is significant only in a small neighborhood of \( k_x = 0 \) for a good beam along the \( y \) axis, and thus \( \gamma \) in Eq. (4) is well approximated by \( k_y/k \) according to Eq. (2), neglecting the third-order terms. A good beam necessarily requires that the fractional change of the beam field along the transverse direction be very small over a wavelength, which itself is of course much smaller than the beam width. As a ready and useful application of Eq. (8), we note that a wave beam field of squarelike (transverse) profile of full width \( 2w \), of unit field amplitude, and centered at the origin has the following partial-wave expression in view of Eq. (3):

\[ E_{\text{sq}} = \sum_{|m|\leq w} J_m(kr)\exp(ik \phi), \]  

where \( kw \) is large, of course.

In passing, we want to point out that the beam-shape coefficient has been evaluated before\(^1\)\(^7\) semiquantitatively, based on the localization principle of van de Hulst.\(^1\)\(^8\) Here we derive it rigorously from first principles under the clearly stated assumption, and the incident wave field need not be Gaussian.

### C. Scattering Efficiency for a Good Beam

For an incident good beam, the incident power \( P_{\text{inc}} \) is just the integration of \( c \varepsilon_0 |E_{\text{inc}}|^2/2 \) along the whole \( x \) axis at \( y = 0 \). Since \( |E_{\text{inc}}(x, 0)| \) varies very little over \( 1/k \), the very definition of integration essentially leads to

\[ \int_{-\infty}^{\infty} dx |E_{\text{inc}}(x, 0)|^2 = \frac{1}{k} \sum_m |E_{\text{inc}}(m/k, 0)|^2. \]  

The scattering efficiency, defined by the ratio of \( P_{\text{sca}} \) to the incident power, can therefore be expressed as

\[ \eta = \frac{4 \sum_m |\beta_m|^2 \sin^2 \delta_m}{\sum_m |\beta_m|^2} \]  

in view of Eq. (8), implying an upper bound of 4 for the quantity. For reference, the scattering efficiency in the case of a plane incident wave is

\[ \sigma_{\text{sca}}/2a = \frac{2}{s} \sum \sin^2 \delta_m, \]  

which has been known in the literature to approach the value of 2 in the limit of \( s \rightarrow \infty \).

### 3. POWER SCATTERED VERSUS POWER INTERCEPTED

We have calculated \( \eta \) in the case of a good beam scattered by a cylinder as a function of the position of the beam relative to the cylinder for a number of values of \( s \) and \( w/a \), assuming always that \( s \gg 1 \) and \( kw \gg 1 \). We first present the computational results, where the beam is taken to be Gaussian of half-width \( w \) and focused at \((x_0, 0)\) with unit amplitude, so that the beam-shape coefficient is

\[ \beta_m = \exp \left[ -\frac{(m/k - x_0)^2}{2w^2} \right] \]  

and the corresponding scattering efficiency is...
A. Computational Results for Narrow Beams \((w \ll a)\)

Three sets of typical graphs for a transparent cylinder of \(n = 1.5\) are plotted in Fig. 2, showing the scattering efficiency \(\eta\) versus the focal position \(x_0\) for three size parameters \(s = 10,000, 10,002,\) and \(10,004,\) respectively, with \(w/a = 0.05, 0.1,\) and \(0.2\) in each set. The bases for the three sets are appropriately chosen to be 0, 2, and 4, respectively.

![Fig. 2. Three sets of graphs showing the scattering efficiency versus the focal position of a Gaussian beam of half-width \(w\) scanning a dielectric cylinder of radius \(a\) for \(s = 10,000, 10,002,\) and \(10,004,\) respectively, with \(w/a = 0.05, 0.1,\) and \(0.2\) in each set. The bases for the three sets are appropriately chosen to be 0, 2, and 4, respectively.](image)

The cylinder can be either a perfect conductor or a transparent dielectric.

B. Computational Results for Broader Beams \((w \gg a)\)

For broader beams scattered by a dielectric cylinder of \(n = 1.5,\) three typical graphs for the scattering efficiency \(\eta\) versus the focal point \(x_0\) of the beam are shown by the solid lines in Fig. 4. Here three half-width values \(w = a, 5a,\) and \(10a\) are taken, and the size parameter is \(s = 10^3.\) Noting that \(\eta = 0\) \((\eta = 2)\) for a narrow beam passing basically outside (striking wholly on) the cylinder, we calculate the ratio of the power directly intercepted by the cylinder to the incident power, i.e.,

\[
P_{\text{itc}}/P_{\text{inc}} = \frac{\int_{-a}^{a} dx \epsilon_0 |E_{\text{inc}}|^2/2P_{\text{inc}}}{\int_{-a}^{a} dx \epsilon_0 |E_{\text{inc}}|^2/2P_{\text{inc}}},
\]

and plot its doubled value versus \(x_0\) (dashed curves) for the three half-widths in the same figure. They are very close to their corresponding solid curves. We have also computed and plotted the graphs for \(s = 10^4;\) the two curves are almost indistinguishable from each other and thus not shown here. All these results clearly show that, for broader beams, the power scattered is always twice the power directly intercepted by the cylinder for a very large size parameter.

![Fig. 4. The three solid curves give the scattering efficiency versus the focal position of a Gaussian beam of half-width \(w\) scanning a dielectric cylinder of radius \(a\) for \(w/a = 1, 5,\) and \(10,\) respectively, with \(s = 10^3.\) The dashed curves give the corresponding \(2P_{\text{itc}}/P_{\text{inc}}\) curves.](image)
C. Analytic Results for Narrow Beams ($w \ll a$)

Here we show that some of the computational results can in fact be obtained analytically for narrow beams as follows, without assuming the actual shape of the beam.

Because the incident beam intensity is significant only in a width $\sim w$ about the focal point $x_0$, the summation in Eq. (11) is basically limited to the range $m$ between $k(x_0 - w)$ and $k(x_0 + w)$. Hence, for the beam passing completely off the cylinder (see Fig. 1), so that $x_0 - w > a$, we have effectively $m > s$ and thus $|J_m(s)/Y_m(s)| \ll 1$ can be well approximated for large $m$. As a result, we have a nearly zero phase shift $\delta_m$ in this range according to Eqs. (6) except perhaps for only one particular $m$ in the case of a transparent cylinder such that $s$ sits exactly on a value of morphology-dependent resonance (whispering-gallery mode). Consequently,

$$\eta \sim 0 \quad \text{for} \quad |x_0| > a + w \quad (16)$$

under the condition $k w > 1$. This exactly shows the first feature of the computational results in Subsection 3.A.

For the beam hitting the cylinder on axis, the summation in Eq. (11) is limited to the range $m \ll s$, so that the Bessel functions can be approximated by their asymptotic forms. If the cylinder is a perfect conductor, Eq. (6b) leads to $\sin^2 \delta_m = \cos^2(s - \pi/4)$ for an even $m$ and $\sin^2 \delta_m = \sin^2(s - \pi/4)$ for an odd $m$, each independent of $m$. Furthermore, because $k w \gg 1$, the sum of the $|\beta_m|^2$ over even $m$ is essentially equal to the sum over odd $m$, each equal to one half of the sum over all $m$. Therefore

$$\eta = 2 \quad \text{for} \quad |x_0| \ll a \quad (17)$$

can be readily obtained for the case of a perfect conductor. If the cylinder is a transparent dielectric, on the other hand, the asymptotic forms can still be applied to Eq. (6a) to yield

$$\sin^2 \delta_m = \frac{(n + 1) \sin[(n - 1)s] + (n - 1) \cos[(n + 1)s]}{2[n^2 + 1 \mp (n^2 - 1) \sin 2ns]} \quad (18)$$

with the upper (lower) sign for an even (odd) $m$, each again independent of $m$. Now it is not difficult to use the same method to find

$$\eta = \frac{2}{2} = 1$$

$$\eta = 2 \left[ 1 - (-1)^p \frac{2n \cos \frac{p \pi}{n}}{n^2 + 1} \right] \quad (19)$$

in the case of a narrow beam hitting the transparent cylinder on axis. This last equation shows the variation of $\eta$ about 2 with respect to $s$. For example, if $s = p \pi/2n$ with a very large integer $p$, we have

$$\eta = 2 \left[ 1 - (-1)^p \frac{2n \cos \frac{p \pi}{n}}{n^2 + 1} \right], \quad (20)$$

obviously implying a substantial variation versus $s$ as $p$ changes from any even (odd) integer to its neighboring odd (even) integers for $n = 1.5$, say. This is in agreement with the third feature of the computational results. In passing, we want to point out that the situation of a narrow beam hitting a large transparent cylinder on axis is analogous to that of a wave incident normally onto a transparent layer of thickness $2a$, where the corresponding "scattered" field is actually the transmitted field minus the incident field in the forward direction and the reflected field in the backward direction. As given in Appendix A, the latter case gives similar oscillations of the "scattering efficiency" versus $s$ and compares well with the results from Eqs. (14) and (19). Thus Lock's oscillation feature is not a surprise. Nevertheless, this points out the danger of taking for granted the conventional $P_{\text{sca}}$ (or $\sigma_{\text{sca}}$) as the actual power scattered.

What we have obtained in this section, together with the already known result for the case of a plane incident wave, clearly shows that, except for the special case of a narrow beam hitting a large cylinder on axis, $P_{\text{sca}}$ is always approximately twice the power directly intercepted by the cylinder for a very large size parameter, irrespective of the waist size and the focal position of the incident beam.

The extinction paradox in the literature concerns the scattering of a plane incident wave. But a plane wave is really a linear superposition of infinitely many beam waves of different focal positions; for example, exp($iky$) is equal to

$$\int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dk_x \exp \left( -\frac{k_x^2 w^2}{2} \right) \exp[i k_x(x - x_0) + ik_y], \quad (21)$$

where the second integral, as is obvious from Eq. (1), is a Gaussian-like beam centered (or focused) at $(x_0, 0)$. Therefore the field scattered by an object of linear properties in the case of a plane incident wave can be considered a linear superposition of all the scattered fields resulting from individual incident beam waves focused at different positions. With this understanding, we immediately see that our results are in direct conflict with the prevailing interpretation of attributing one half of the "scattering efficiency" to edge diffraction because this requires deflection of incident rays at an impact parameter as far as $2a$, twice the radius from the axis even for $a \gg \lambda$! The parts of the plane incident wave passing off the physical boundary simply do not interact with the object and thus make no contribution to $P_{\text{sca}}$ or $\sigma_{\text{sca}}$ in the limit of $\lambda/a \to 0$, in agreement with the notion of geometrical optics. Besides, a narrow beam hitting well within the physical area of the scatterer already gives a value of $P_{\text{sca}}$ twice that of the power intercepted, yet there is no part of the beam passing off the surface, let alone edge diffraction.

4. ACTUAL POWER SCATTERED

As pointed out in Section 1, the paradox has been explained before by realizing that the total field outside is a superposition of the incident field and the scattered field, which consists of a reflection (or deflection) and another SFW part, the latter being responsible for forming the...
shadow right behind the object. Here we want to study the actual power scattered by considering an incident wave beam, using the 2D partial-wave analysis that we just introduced. The size parameter \(s (= k a)\) has to be very large, and, for simplicity, we shall restrict the discussion to a perfectly conducting cylinder. In passing, we note that the scattering of a plane wave by a totally reflecting sphere has been treated in some detail by Nussenzveig.\(^{19}\)

### A. Narrow-Beam Case \((w < a)\)

First, consider a narrow beam \((w < a)\) wholly striking the cylinder as sketched in Fig. 5, where the dashed curves show the further path of the incident beam in the absence of the scatterer. By expressing the Bessel function in Eq. (3) in terms of the two Hankel functions, we have

\[
E_{\text{inc}}(x, y) = \sum_{m} \beta_m \left( \frac{H^+_m(kr) + H^-_m(kr)}{2} \right) \exp(i m \phi),
\]

(22)

where \(H^+_m\) is the Hankel function with incoming wave at infinity. We now claim that, in the region where \(kr \gg m\), the first sum [with the \(H^+_m\)] alone gives the incident beam field in the forward region satisfying \(0 < \phi < \pi\), whereas the second sum [with the \(H^-_m\)] alone gives the incident beam in the backward region satisfying \(0 > \phi > -\pi\). (More precisely, the first (second) sum gives the incident beam in the space between the two parallel dashed curves for \(y > 0\) (\(<0\) outside the cylinder in Fig. 5.) Physically, this makes good sense because the forward (backward) region is where the incident wave is considered outgoing (incoming). Mathematically, this can be shown by noting that, for a beam interacting with the cylinder, the effective range of \(m\) is from \(k(x_0 - w)\) to \(k(x_0 + w)\) and that therefore the Hankel functions can be well approximated by their asymptotic forms well outside the cylinder. The second (first) sum can thus be shown to be negligible in the forward (backward) region because \(\phi + \pi/2\) (\(\phi - \pi/2\)) is always well away from 0, so that all the terms in the sum can be considered points in the complex plane evenly distributed around the origin with slowly varying radius.

On the other hand, the scattered field in Eq. (5) can be cast as

\[
E_{\text{sc}}(x, y) = \frac{1}{2} \sum_{m} \beta_m \left( \exp(2i m \delta_m) - 1 \right) \frac{H^+_m(kr)}{2} \exp(i m \phi)
\]

(23)

if the sine function is expressed in terms of the exponential functions. By comparing Eq. (23) with the previous equation, we immediately see that the second term, which is exactly opposite the incident beam field in the forward region, is just the SPW and it is beamlike. (This wave is dominant only in the space between the two dashed curves in Fig. 5 outside the cylinder.) As a result, the “actual” scattered field, which is obviously the outgoing-wave part of the total field \(E_{\text{inc}} + E_{\text{sc}}\), is given by

\[
E_{\text{sc}}(x, y) = \frac{1}{2} \sum_{m} \beta_m \left( \frac{H^+_m(kr) + H^-_m(kr)}{2} \right) \exp(i m \phi),
\]

(24)

immediately leading to

\[
P_{\text{sc}} = \frac{1}{2 \mu_0 \omega} \sum_{m} |\beta_m|^2
\]

(25)

as the actual power scattered. In view of relation (10), it is exactly equal to the incident power striking the cylinder. This result is certainly expected and consistent with geometrical optics. There is no paradox at all. In passing, we note that the field in Eq. (24) is just the reflection (or deflection) part of the scattered field, for which, for an incident plane wave (or a very broad incident beam), \(m \ll 1\) holds and \(\beta_m = 1\).

### B. Very-Broad-Beam Case \((w \gg a)\)

Next, consider a very broad beam interacting with the cylinder on axis, i.e., \(x_0 = 0\). Likewise, we express the Bessel function in Eq. (3) in terms of the two Hankel functions, and, in addition, we break the sum into two parts, i.e.,

\[
E_{\text{inc}} = \left( \sum_{|m| \ll s} + \sum_{|m| > s} \right) \times \beta_m \left[ \frac{H^+_m(kr) + H^-_m(kr)}{2} \right] \exp(i m \phi).
\]

(26)

Compared with relation (9), the first partial sum, where \(\beta_m = 1\) can be taken for \(|m| \ll s\), is just the squarelike-profile beam field \(E_{\text{eq}}\) of full waist width \(2a\) propagating along the \(y\) axis, whose first part with the \(H^+_m\) is just the squarelike-profile beam field in the forward region satisfying \(kr \gg s\).

The scattered wave given by Eq. (5), on the other hand, can be simplified by limiting the sum to only the values of \(|m| \ll s\) because, as shown above, \(\sin \delta_m = 0\) for \(|m| > s \gg 1\). (This has the obvious physical meaning that the scattered field cannot have angular momentum large compared with \(ka\).) It is then easy to see that

![Fig. 5. Sketch of a collimated yet narrow incident wave beam wholly striking the (circular) cylinder, the dashed curves showing the further path of the beam in the absence of the scatterer.](image-url)
\[ E_{\text{sca}} = \frac{1}{2} \sum_{|m| < s} \left[ \exp(2i\delta_m) - 1 \right] H_m^{(1)}(kr) \exp(im\phi), \]  

where again \( \beta_m = 1 \) has been taken. The second term, when compared with the first part of the first partial sum in Eq. (26), gives a square-profile beam field of the same strength but of opposite sign in the forward region behind the cylinder. This is the SFW responsible for the shadow behind the cylinder well within the Rayleigh range \( ka^2 \) of the beam. Beyond the Rayleigh range, the Hankel function takes its asymptotic form, and the SFW becomes

\[ E_{st} = -\frac{1}{2} \left( \frac{2}{\pi kr} \right)^{1/2} \exp\left( ikr - \frac{i\pi}{4} \right) \sum_{m=-s}^{s} \exp(im\phi), \]  

where we have introduced the deviation angle \( \psi = \phi - \pi/2 \) from the forward \( y \) axis. The sum can be easily obtained in view of the terms forming a geometric progression. For very large \( s \), it yields

\[ E_{st} = -A_\delta \exp\left( ikr - \frac{i\pi}{4} \right) \]  

for small \( \psi \), where

\[ A_\delta = \left( \frac{2}{\pi kr} \right)^{1/2} \frac{\sin(ka\psi)}{\psi}, \]  

which, for a given \( r \), is a sinc function peaked at \( \psi = 0 \) and oscillating with a decreasing amplitude on either side. Obviously, \( A_\delta^2 \) gives the well-known single-slit diffraction pattern, the integration of which over a circular arc in the forward direction gives \( 2\pi I_\delta \) as the total outgoing power, which is equal to the incident power through the slit as expected.

A well-known result, which is of importance here, is that the beam wave diverges with a divergence angle \( 2\psi_0 \) of the order of \( 1/ka \) (i.e., \( 1/a \)) beyond the Rayleigh range \( ka^2 \) of the SFW beam. Because the incident beam has a much larger waist, its divergence angle \( 2\psi_w \), say) is much smaller than \( 2\psi_0 \). Consequently, the SFW eventually diverges out of the incident beam. The situation is shown in Fig. 6, where the SFW beam in four regions behind the cylinder is sketched by the dashed lines together with the very broad incident beam represented by the two long parallel solid lines (their separation is not to scale). The SFW beam retains its width in the first region (at the bottom), starts to diverge in the second region \( (r > sa) \) at the angle \( 2\psi_0 \) but is still inside the incident beam, keeps diverging in the third region \( (kw^2 > r > sw) \) so that it is partly outside the incident beam, and eventually in the fourth region \( (kw^2 > r >> sw) \) and beyond spreads almost completely outside the incident beam. This is exactly what was meant by Peierls.10 The actual power scattered is indeed twice the power intercepted. But this is true only for a very broad incident beam.

An incident beam of waist slightly larger than the cylinder size has itself a divergence angle that is slightly smaller than that of the SFW beam. The latter can thus never get away from the incident beam. As a result, the actual power scattered should lie basically between \( P_{\text{sc}} \) and \( 2P_{\text{sc}} \), depending on how large the incident beam waist is compared with the cylinder size. Section 5 will treat this in more detail.

5. POWER DEPLETED FROM THE INCIDENT WAVE

From energy conservation, the actual power scattered is the power depleted from the incident wave. In fact, it is the latter quantity that is usually measured in optical experiments. Consider, for simplicity, a wave field of the squarelike profile given by relation (9), incident onto the cylinder as sketched in Fig. 6. (The half-width \( w \) now need not be much larger than \( a \).) If one measures the power with a light collector of width slightly larger than \( 2a \) along the forward \( y \) axis at a distance \( a < r < ka \) beyond the cylinder, one should obtain \( P_{\text{sc}} \) as the power depleted because the SFW cancels well the incident wave in the region and the reflected rays are largely uncollected. This result holds irrespective of the size of the incident beam width and is in total agreement with geometrical optics.

Yet scattering includes the contribution from asymptotically outgoing waves at large distance, which is outside the realm of geometrical optics. We have to consider the region where \( r > kw^2 \), especially when \( w \sim a \). Here the incident wave and the SFW are both of the form of Eq. (29), with, of course, \( a \) replaced by \( w \) and a positive sign in front for the incident wave. The actual field there, being equal to the sum of the incident field and the SFW field, yields the following intensity variation (in units of the incident intensity \( I_0 \) encountered by the cylinder):

\[
\frac{I(r, \psi)}{I_0} = \frac{2}{\pi kr} \left( \frac{\sin^2 kw\psi}{\psi^2} + \frac{\sin^2 ka\psi}{\psi^2} \right) - 2 \frac{\sin kw\psi \sin ka\psi}{\psi^2},
\]  

\[ \text{Fig. 6. Sketch of the SFW beam, indicated by the dashed lines, in four regions behind the (circular) cylinder as a result of interaction with a very broad incident beam, indicated by the two parallel long solid lines (width not to scale). The bulk of the SFW expands completely out of the incident beam in the top region.} \]
where the third term is due to interference. In view of 

\[ k \alpha > 1, \]

the arc-length integration of the three terms over a
finite angle about the forward direction can be shown to
give \( 2w, \) \( 2a, \) and \(-4a,\) respectively.\(^2\) Together with an-
other \( 2a \) contributed by the reflection (or deflection) part
of the scattered wave, the result is simply a statement of
energy conservation. In the literature, the negative re-
sult due to interference has been identified with the
power depleted from the incident wave.\(^7,16\) This iden-
tification is legitimate in the case of a very broad incident
beam, where the energy carried by the second term has
diverged completely out of the incident wave well beyond
the Rayleigh range of the SFW beam. Yet it is not valid
for other cases, in view of the discussion already given in
Section 4. The crux of the matter is that, for an incident
beam of waist slightly or moderately larger than the di-
ameter of the cylinder, the SFW is always mixed with the
incident wave to some extent and can never get com-
pletely away from it even at infinity.

A simple and sensible way to quantitatively estimate
the actual power depleted from the incident wave of any
beam width is to examine the arc-length integration of
Eq. (31) over the angular interval \((-N \pi/kw, N \pi/kw),\)
say, with \( N \) being an integer. This is equivalent to saying
that a light collector of that angular width is being used.
Obviously, in the absence of the scatterer, where only
the first term survives, the integration gives the bulk of
the incident wave power. (In fact, it gives approximately
90%, 95%, 97%,... of the total incident power for \( N =
1,2,3,...,\) respectively.) Therefore, with the cylinder in
place, the power depleted, \( P_{\text{dep}}, \) from the incident wave
comes from the arc-length integration of the latter two
terms. The result is

\[
P_{\text{dep}} = 2a \int_{-N \pi/a}^{N \pi/a} \mathrm{d} \xi \left( 2 \sin \frac{w \xi}{a} - \sin \xi \right) \frac{\sin \xi}{\pi \xi^2},
\]

where we have changed the integration variable to \( \xi = k \alpha \psi. \)
This is essentially the actual scattering “cross section.”
For a very broad incident beam \( (w/a \gg N \gg 1), \) the first term gives approximately a value of \( 4a, \)
whereas the second term makes a negligible contribution,
reflecting the fact that the SFW beam by itself has
diverged essentially out of the incident wave; it is easy to
see that the first term, though approaching \( 4a \) for \( N \gg
1, \) always gives a value slightly larger (smaller) than
the limit for an odd (even) \( N. \) For \( w = a, \) on the other
hand, the result is essentially \( 2a. \) For any \( w/a \gg 1, \)
the integration can in fact be carried out to yield

\[
P_{\text{dep}} = \frac{4a}{\pi} \left[ \left( 1 + \frac{w}{a} \right) \text{si} \left( N \pi \left( 1 + \frac{a}{w} \right) \right) + \left( 1 - \frac{w}{a} \right) \text{si} \left( N \pi \left( 1 - \frac{a}{w} \right) \right) - \text{si} \left( \frac{2N \pi a}{w} \right) \right]
+ \frac{\sin^2 \left( N \pi a/w \right)}{N \pi a/w},
\]

where \( \text{si}(x) \) is the sine-integral function defined by

We show in Fig. 7 the variation of \( P_{\text{dep}}/I_0 \) as a function of
\( w/a \) according to Eq. (33) for \( w/a \gg 1. \) The dotted,
dashed, and solid curves are for \( N = 1, 2, \) and \( 3, \) respec-
tively. Though expectedly with slight differences, they
have the same general features, namely, taking the value
of \( ~2a \) at \( w = a, \) becoming basically larger for a broader
beam, and approaching the value of \( -4a \) for \( w \gg a. \) For
\( w/a \approx 1, \) a line is simply drawn connecting the origin and
the point (1, 2a); this is a consequence of Section 4. The
inset shows the finer variation for smaller values of \( w/a, \)
where we note that the solid curve for \( N = 3 \) connects
quite smoothly to the line for \( w/a \approx 1. \)

6. DISCUSSION AND CONCLUSIONS

Through the scattering of a 2D wave beam by a trans-
verse absorptionless cylinder and the application of
partial-wave analysis to the case of a large size parameter
\( (s = k a \gg 1), \) we have obtained several results regarding
the extinction paradox. First, we have shown that the
conventional power scattered, \( P_{\text{sca}} \) (or, equivalently, the
conventional scattering cross section \( \sigma_{\text{sca}}), \) except for
a very special case, is always twice the power directly inter-
cepted by the scatterer, \( P_{\text{itc}}, \) irrespective of the waist size
and the position of the incident beam. This rules out the
existing interpretation of attributing one half of \( P_{\text{sca}} \) to
edge diffraction.

It is important to recognize that the scattered field
\( E_{\text{sca}}, \) which is used to calculate \( P_{\text{sca}}, \) is not the total field.
This recognition naturally leads to the understanding of
\( E_{\text{sca}} \) as consisting of two parts, the reflection part and the
SFW part, each contributing an equal amount of outgoing
wave energy for short wavelengths, leading to the result
\( P_{\text{sca}} = 2P_{\text{itc}}. \) The SFW part, which is dominant in
the forward direction, leads us further to show that the actual
power scattered (i.e., the power depleted from the inci-
dent wave) is equal to one unit of \( P_{\text{itc}} \) for a narrow inci-
dent beam \( (w \ll a), \) increases in general for a broader
beam \( (w \gg a), \) and eventually becomes \( 2P_{\text{itc}} \) for a very
broad beam \((w \gg a)\). The latter agrees with what is being accepted in the literature, where a plane incident wave or a much broader incident beam is always assumed. Yet the interpretation is subtle. The amount in excess of one unit of \(P_{\text{inc}}\) is due to the intrinsic eventual divergence of the SFW beam. Its value depends basically on how much the SFW beam diverges \textit{out} of the incident beam well beyond the Rayleigh range, and only at this large distance is the actual power scattered measurable. Moreover, while the SFW beam by itself has diverged, its influence still remains in the forward direction through interference. It is exactly this interference that weakens the incident wave intensity to guarantee energy conservation.

In conclusion, several further remarks are in order. First, the special case (i.e., Lock’s case) mentioned in the first paragraph of this section can again be understood by recognizing \(E_{\text{sec}}\) as not being the total field and by noting the similarity to the situation of transmission through a transparent layer. Second, there is no reason to worry about geometrical optics because the excess amount of the actual power scattered is really due to the divergence of the SFW beam well beyond its Rayleigh range, where the rectilinear propagation of a pencil of light no longer holds.\(^{21}\) In this connection, we might as well point out that Brillouin, who excluded the forward angular interval in the calculation of his modified scattering cross section, was apparently unaware of this possible contribution from the SFW beam as asymptotically outgoing wave at far distance. Finally, the power depleted from the incident wave, obtained in Section 5, should still be valid for cases other than that of a totally reflecting cylinder. This is because the analysis involves only the incident wave and the SFW beam wave, both having the same form irrespective of the property of the cylinder.

\textbf{APPENDIX A: OSCILLATING “SCATTERING EFFICIENCY” ABOUT 2 VERSUS \(s\)}

A narrow beam hitting a very large transparent cylinder on axis is analogous to that of a wave incident normally onto a dielectric layer of thickness \(2a\), where the corresponding “scattered” field is actually the transmitted field minus the incident field in the forward direction and the reflected field in the backward direction. If the well-known transmission and reflection coefficients\(^{22}\) are used, the corresponding “efficiency” is

\[
\eta = 2 - 2 \times \frac{2n[(n^2 + 1)\sin 2ns \sin 2s + 2n \cos 2ns \cos 2s]}{(n^2 + 1)^2 \sin^2 2ns + 4n^2 \cos^2 2ns}.
\]

\textit{(A1)}

In Fig. 8, we take \(n = 1.5\) and plot the result (dotted curve) versus \(s\) between 10,000 and 10,020. For comparison, we also plot the asymptotic result (dashed curve) according to Eq. (19) and the numerical results according to Eq. (14) with \(x_0 = 0\), represented by the three solid curves 1, 2, and 3 for \(w/a = 0.005, 0.02,\) and 0.05, respectively. The dotted and dashed curves compare well with the solid curve for the narrowest beam.

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