Pricing multi-asset financial derivatives with time-dependent parameters — Lie algebraic approach

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Abstract

In this paper we present a Lie-algebraic technique for the valuation of multi-asset financial derivatives with time-dependent parameters. By exploiting the dynamical symmetry of the pricing partial differential equations of the financial derivatives, the new method enables us to derive analytical closed-form pricing formulae very straightforwardly. We believe that this new approach will provide an efficient and easy-to-use method for the valuation of financial derivatives.

1. Introduction

The Lie-algebraic method is introduced by Lo and Hui (2001) to the field of finance for the pricing of single-asset financial derivatives with time-dependent model parameters. This new method is based upon the Wei-Norman theorem (Wei and Norman, 1963) and has never been used in the field of finance. It is very simple and has been successfully applied to tackle time-dependent Schrödinger equation associated with generalized quantum time-dependent oscillators (Lo, 1993a and 1993b; Ng and Lo, 1997; and Lo and Wong, 1995) as well as the Fokker-Planck equation (Lo, 1997, 1998a, 1998b and 1998c). By exploiting the well-defined algebraic structures of the pricing partial differential equations, analytical closed-form pricing formulae can be derived for financial derivatives with time-dependent parameters. For demonstration, we have applied the Lie-algebraic approach to value European options for the constant elasticity of variance (CEV) process and corporate discount bonds with default risk. In this paper we shall extend the Lie-algebraic approach to the valuation of financial derivatives involving multi-assets and stochastic interest rate, e.g. multi-asset options with and without stochastic short-term interest rate. In the valuation of these financial derivatives the value of each of the underlying assets is assumed to follow the usual lognormal diffusion process:

$$\frac{dS_i}{S_i} = \mu_i(t)dt + \sigma_i(t)dZ_i \quad , \qquad 1 \leq i \leq N \tag{1}$$

where $\mu_i(t)$ and $\sigma_i(t)$ are the drift and volatility of the value of asset i, respectively. The dynamics of the short-term interest rate r is drawn from the term structure model (Vasicek, 1977):

$$dr = \kappa(t)\left[\theta(t) - r\right]dt + \sigma_r(t)dZ_r \quad , \tag{2}$$

where the short-term interest rate is mean-reverting to long-term mean $\theta(t)$ at speed $\kappa(t)$ and $\sigma_r(t)$ is the volatility of r. The Wiener processes dZ_r and dZ_i are correlated with

$$dZ_i dZ_r = \rho_{ir}(t)dt \qquad , \qquad dZ_i dZ_j = \rho_{ij}(t)dt \quad , \tag{3}$$

where $\rho_{ir}(t)$ and $\rho_{ij}(t)$ are the correlation coefficients, and we must necessarily have $\rho_{rr}(t) = \rho_{ii}(t) = 1$, $-1 < \rho_{ir}(t) = \rho_{ri}(t) < 1$ and $-1 < \rho_{ij}(t) = \rho_{ji}(t) < 1$ for $1 \leq i, j \leq N$. It has been pointed out that such a pricing problem is rather formidable and defies the conventional approach for the single-asset Black-Scholes model with time-dependent parameters (Bos and Ware, 2000). Nevertheless, within the framework of the Lie-algebraic approach, the generalization is very simple and straightforward.

This paper is organized as follows. Section 2 outlines the Wei-Norman theorem and its applications. Section 3 applies the Lie-algebraic technique to the valuation problem of multi-asset options, in which the short-term interest rate is not treated as a stochastic variable. Section 4 studies the pricing of multi-asset options with stochastic short-term interest rate using the new valuation approach. Finally, section 5 briefly summaries and concludes the paper.

2. Wei-Norman Theorem

Consider the linear operator differential equation of the first order

$$\frac{dU(t)}{dt} = H(t)U(t) \qquad ; \qquad U(0) = 1 \tag{4}$$

where H and U are both time-dependent linear operators in a Banach space or a finite-dimensional space. According to the Wei-Norman theorem (Wei and Norman, 1963), if the operator H can be expressed as

$$H(t) = \sum_{n=1}^{N} a_n(t) L_n \quad , \tag{5}$$

where a_n's are scalar functions of time and L_n are the generators of an N-dimensional solvable Lie algebra or the real split 3-dimensional simple Lie algebra, then the operator U can be expressed as

$$U(t) = \prod_{n=1}^{N} \exp\left[g_n(t) L_n\right] \quad . \tag{6}$$

Here the g_n's are time-dependent scalar functions to be determined. To find the g_n's, we simply substitute Eq.(5) and Eq.(6) into Eq.(4), and compare the two sides term by term to obtain a set of coupled nonlinear differential equations

$$\frac{dg_n(t)}{dt} = \sum_{m=1}^{N} \eta_{nm} a_m(t) \qquad , \qquad g_n(0) = 0 \tag{7}$$

where η_{nm} are nonlinear functions of g_n's. Thus, we have reduced the linear operator differential equation, Eq.(4), to a set of coupled nonlinear differential equations of scalar functions, Eq.(7).

For illustration, we consider the special case that the generators L_n's form the Heisenberg-Weyl Lie algebra defined by the commutation relations:

$$[L_1, L_2] = L_3 \qquad , \qquad [L_1, L_3] = [L_2, L_3] = 0 \quad . \tag{8}$$

Then H is given by

$$H(t) = a_1(t)L_1 + a_2(t)L_2 + a_3(t)L_3 \quad . \tag{9}$$

According to the Wei-Norman theorem, $U(t)$ can be expressed as

$$U(t) = \exp\left[g_1(t)L_1\right] \cdot \exp\left[g_2(t)L_2\right] \cdot \exp\left[g_3(t)L_3\right] \quad . \tag{10}$$

By differentiation, we obtain

$$\begin{aligned} \frac{dU(t)}{dt}U(t)^{-1} &= \frac{dg_1(t)}{dt}L_1 + \frac{dg_2(t)}{dt}\exp\left[g_1(t)L_1\right]L_2\exp\left[-g_1(t)L_1\right] + \\ &\quad \frac{dg_3(t)}{dt}\exp\left[g_1(t)L_1\right]\exp\left[g_2(t)L_2\right]L_3\exp\left[-g_2(t)L_2\right]\exp\left[-g_1(t)L_1\right] \\ &= \frac{dg_1(t)}{dt}L_1 + \frac{dg_2(t)}{dt}L_2 + \left[\frac{dg_3(t)}{dt} + g_1(t)\frac{dg_2(t)}{dt}\right]L_3 \quad . \end{aligned} \tag{11}$$

Comparing Eq.(9) and Eq.(11) yields a set of three coupled nonlinear differential equations:

$$\begin{aligned} \frac{dg_1(t)}{dt} &= a_1(t) \\ \frac{dg_2(t)}{dt} &= a_2(t) \\ \frac{dg_3(t)}{dt} + g_1(t)\frac{dg_2(t)}{dt} &= a_3(t) \quad . \end{aligned} \tag{12}$$

It is not difficult to show that the set of differential equations can be easily solved by quadrature:

$$\begin{aligned} g_1(t) &= \int_0^t d\tau a_1(\tau) \\ g_2(t) &= \int_0^t d\tau a_2(\tau) \\ g_3(t) &= \int_0^t d\tau \left[a_3(\tau) - a_2(\tau) g_1(\tau)\right] \quad . \end{aligned} \tag{13}$$

As a result, the operator $U(t)$ is thus determined.

3. Multi-asset European options

The fair price $P(S_1, S_2, ..., S_n, t)$ of a multi-asset European option with time-dependent parameters can be determined by solving the multi-asset generalization of the Black-Scholes equation

$$\frac{\partial P}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{n} \sigma_i(t)\sigma_j(t)\rho_{ij}(t) S_i S_j \frac{\partial^2 P}{\partial S_i \partial S_j} + \sum_{i=1}^{n} \left[r(t) - d_i(t)\right] S_i \frac{\partial P}{\partial S_i} - r(t) P \quad , \tag{14}$$

where t is the time to maturity. Introducing the new variables $x_i = \ln(S_i)$, the pricing equation is simplified to

$$\begin{aligned} \frac{\partial P}{\partial t} &= \frac{1}{2} \sum_{i,j=1}^{n} \sigma_i(t)\sigma_j(t)\rho_{ij}(t) \frac{\partial^2 P}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \left[r(t) - d_i(t) - \frac{\sigma_i(t)^2}{2}\right] \frac{\partial P}{\partial x_i} - r(t) P \\ &\equiv \left[H(t) - r(t)\right] P \quad . \end{aligned} \tag{15}$$

It is obvious that the operator $H(t)$ can be rewritten as follows:

$$H(t) = \sum_{i,j=1}^{n} A_{ij}(t) \hat{L}_{ij} + \sum_{i=1}^{n} B_i(t) \hat{D}_i \quad , \tag{16}$$

where

$$\begin{aligned} \hat{L}_{ij} &= \frac{\partial^2}{\partial x_i \partial x_j} \qquad , \qquad \hat{D}_i = \frac{\partial}{\partial x_i} \\ A_{ij}(t) &= \frac{1}{2}\sigma_i(t)\sigma_j(t)\rho_{ij}(t) \qquad , \qquad B_i(t) = r(t) - d_i(t) - \frac{\sigma_i^2}{2} \quad . \end{aligned} \tag{17}$$

The operators L_{ij} and D_i form a solvable algebra; in fact, they all commute. We may now define the evolution operator $U(t,0)$ such that

$$P(x_1, x_2, ..., x_n, t) = \exp\left[-\int_0^t r(t')\ dt'\right] \cdot U(t,0) P(x_1, x_2, ..., x_n, 0) \quad . \tag{18}$$

Inserting Eq.(18) into Eq.(15) yields the evolution equation

$$\frac{\partial}{\partial t} U(t,0) = H(t) U(t,0) \qquad , \qquad U(0,0) = 1 \quad . \tag{19}$$

Since the operators L_{ij} and D_i all commute with each other, the Wei-Norman theorem states that the evolution operator $U(t,0)$ can be expressed in the form (Wei and Norman, 1963)

$$U(t,0) = \prod_{i=1}^{n} \exp\left[b_i(t) \hat{D}_i\right] \cdot \prod_{i,j=1}^{n} \exp\left[a_{ij}(t) \hat{L}_{ij}\right] \quad , \tag{20}$$

where the coefficients $a_{ij}(t)$ and $b_i(t)$ are simply given by

$$\begin{aligned} a_{ij}(t) &= \frac{1}{2} \int_0^t \sigma_i(t') \sigma_j(t') \rho_{ij}(t')\ dt' \\ b_i(t) &= \int_0^t \left[r(t') - d_i(t') - \frac{\sigma_i(t')^2}{2} \right]\ dt' \quad . \end{aligned} \tag{21}$$

Hence, we have found an exact form of the time evolution operator $U(t,0)$.

We define $\mathbf{a}(t)$ as the $n \times n$ matrix whose elements are given by $a_{ij}(t)$, and $\mathbf{a^{-1}}(t)$ as its inverse. Then it is not difficult to show that

$$\begin{aligned} P(x_1, x_2, ..., x_n, t) &= \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \cdots \int_{-\infty}^{\infty} dy_n\ P(y_1, y_2, ..., y_n, 0) \\ &\quad \times K(x_1, x_2, ..., x_n, t; y_1, y_2, ..., y_n, 0) \end{aligned} \tag{22}$$

where

$$\begin{aligned} & K(x_1, x_2, ..., x_n, t; y_1, y_2, ..., y_n, 0) \\ = & \frac{1}{\sqrt{(4\pi)^n \det(\mathbf{a})}} \exp\left[-\int_0^t r(t')\ dt'\right] \times \\ & \exp\left\{ -\frac{1}{4} \sum_{i,j=1}^{n} (x_i - y_i + b_i)(\mathbf{a^{-1}})_{ij} (x_j - y_j + b_j) \right\} \end{aligned} \tag{23}$$

is the propagator of the pricing equation in Eq.(15). With $n = 1$, we shall recover the well-known result of single-asset option pricing.

4. Multi-asset European options with stochastic interest rate

In the presence of stochastic short-term interest rate, the price $P(S_1, S_2, ..., S_n, r, t)$ of a multi-asset European option obeys the partial differential equation

$$\begin{aligned}\frac{\partial P}{\partial t} &= \frac{1}{2}\sum_{i,j=1}^{n}\sigma_i(t)\sigma_j(t)\rho_{ij}(t)S_iS_j\frac{\partial^2 P}{\partial S_i\partial S_j} + \frac{1}{2}\sigma_r(t)^2\frac{\partial^2 P}{\partial r^2} + \\ &\sum_{i=1}^{n}\sigma_i(t)\sigma_r(t)\rho_{ir}(t)S_i\frac{\partial^2 P}{\partial S_i\partial r} + \sum_{i=1}^{n}\left[r - d_i(t)\right]S_i\frac{\partial P}{\partial S_i} + \\ &\kappa(t)\left[\theta(t) - r\right]\frac{\partial P}{\partial r} - rP \\ &= \frac{1}{2}\sum_{i,j=1}^{n}\sigma_i(t)\sigma_j(t)\rho_{ij}(t)\frac{\partial^2 P}{\partial x_i\partial x_j} + \frac{1}{2}\sigma_r(t)^2\frac{\partial^2 P}{\partial r^2} + \\ &\sum_{i=1}^{n}\sigma_i(t)\sigma_r(t)\rho_{ir}(t)\frac{\partial^2 P}{\partial x_i\partial r} + \sum_{i=1}^{n}\left[r - d_i(t) - \frac{1}{2}\sigma_i(t)^2\right]\frac{\partial P}{\partial x_i} + \\ &\kappa(t)\left[\theta(t) - r\right]\frac{\partial P}{\partial r} - rP \quad ,\end{aligned} \tag{24}$$

where $x_i = \ln(S_i)$ and t is the time to maturity. To solve this partial differential equation, we first define the evolution operator $U(t,0) \equiv U_0(t,0)U_I(t,0)$ such that

$$\begin{aligned}P(x_1, x_2, ..., x_n, r, t) &= U(t,0)P(x_1, x_2, ..., x_n, r, 0) \\ &= U_0(t,0)U_I(t,0)P(x_1, x_2, ..., x_n, r, 0) \quad .\end{aligned} \tag{25}$$

Inserting Eq.(25) into Eq.(24) yields the evolution equations

$$H_0(t)U_0(t,0) = \frac{\partial}{\partial t}U_0(t,0) \quad , \quad U_0(0,0) = 1 \tag{26}$$

$$H_I(t)U_I(t,0) = \frac{\partial}{\partial t}U_I(t,0) \quad , \quad U_I(0,0) = 1 \tag{27}$$

where

$$\begin{aligned}H_0(t) &= \frac{1}{2}\sum_{i,j=1}^{n}\sigma_i(t)\sigma_j(t)\rho_{ij}(t)\frac{\partial^2}{\partial x_i\partial x_j} + \frac{1}{2}\sigma_r(t)^2\frac{\partial^2}{\partial r^2} + \\ &\sum_{i=1}^{n}\sigma_i(t)\sigma_r(t)\rho_{ir}(t)\frac{\partial^2}{\partial x_i\partial r} + \sum_{i=1}^{n} r\frac{\partial}{\partial x_i} - \kappa(t)r\frac{\partial}{\partial r}\end{aligned} \tag{28}$$

and $H_I \equiv U_0(t,0)^{-1}\left[H(t) - H_0(t)\right]U_0(t,0)$. It is not difficult to show that the operator $H_0(t)$ can be rewritten in the following form:

$$H_0(t) = \sum_{i,j=1}^{n}A_{ij}(t)\hat{L}_{ij} + \sum_{i=1}^{n}E_i(t)\hat{D}_i + \sum_{i=1}^{n}F_i(t)\hat{M}_i + B_1\hat{J}_1 + B_2\hat{J}_2$$

where

$$
\begin{aligned}
\hat{L}_{ij} &= \frac{\partial^2}{\partial x_i \partial x_j} \quad , \quad \hat{D}_i = r\frac{\partial}{\partial x_i} \quad , \quad \hat{M}_i = \frac{\partial^2}{\partial x_i \partial r} \quad , \\
\hat{J}_2 &= \frac{\partial^2}{\partial r^2} \quad , \quad \hat{J}_1 = r\frac{\partial}{\partial r} \quad , \\
A_{ij}(t) &= \frac{1}{2}\sigma_i(t)\sigma_j(t)\rho_{ij}(t) \quad , \quad B_1(t) = -\kappa(t) \quad , \quad B_2(t) = \frac{1}{2}\sigma_r(t)^2 \quad , \\
E_i(t) &= 1 \quad , \quad F_i(t) = \sigma_i(t)\sigma_r(t)\rho_{ir}(t) \quad . \qquad (29)
\end{aligned}
$$

The operators $\hat{L}_{ij}$, $\hat{D}_i$, $\hat{M}_i$ and $\hat{J}_i$ form a solvable Lie algebra:

$$
\begin{aligned}
\left[\hat{L}_{ij}, \hat{L}_{kl}\right] &= \left[\hat{L}_{ij}, \hat{D}_k\right] = \left[\hat{L}_{ij}, \hat{M}_k\right] = \left[\hat{L}_{ij}, \hat{J}_1\right] = \left[\hat{L}_{ij}, \hat{J}_2\right] = \left[\hat{M}_i, \hat{J}_2\right] = 0 \\
\left[\hat{D}_i, \hat{M}_j\right] &= -\hat{L}_{ij} \quad , \quad \left[\hat{D}_i, \hat{J}_1\right] = -\hat{D}_i \quad , \quad \left[\hat{D}_i, \hat{J}_2\right] = -2\hat{M}_i \quad , \\
\left[\hat{M}_i, \hat{J}_1\right] &= \hat{M}_i \quad , \quad \left[\hat{J}_1, \hat{J}_2\right] = -2\hat{J}_2 \qquad (30)
\end{aligned}
$$

where $i, j, k, l = 1, 2, 3, ..., n$. According to the Wei-Norman theorem (Wei and Norman, 1963), the evolution operator $U_0(t,0)$ can be expressed in the form

$$
\begin{aligned}
U_0(t,0) &= \exp\left[\sum_{i=1}^{n} b_i(t)\hat{D}_i\right] \exp\left[\sum_{i,j=1}^{n} a_{ij}(t)\hat{L}_{ij}\right] \exp\left[c_2(t)\hat{J}_2\right] \times \\
& \exp\left[\sum_{i=1}^{n} f_i(t)\hat{M}_i\right] \exp\left[c_1(t)\hat{J}_1\right] \qquad (31)
\end{aligned}
$$

where the coefficients $a_{ij}(t)$, $c_i(t)$, $b_i(t)$ and $f_i(t)$ are to be determined. Then by direct differentiation with respect to t, we obtain

$$
\begin{aligned}
\frac{\partial U_0(t,0)}{\partial t} U_0(t,0)^{-1} &= \sum_{i,j=1}^{n} g_{ij}(t)\hat{L}_{ij} + \sum_{i=1}^{n} h_i(t)\hat{D}_i + \sum_{i=1}^{n} p_i(t)\hat{M}_i \\
& + q_1(t)\hat{J}_1 + q_2(t)\hat{J}_2 \qquad (32)
\end{aligned}
$$

with

$$
\begin{aligned}
g_{ij}(t) &= \frac{\partial a_{ij}}{\partial t} - b_j\frac{\partial f_i}{\partial t} + b_i b_j \frac{\partial c_2}{\partial t} + b_j\left(2c_2 b_i - f_i\right)\frac{\partial c_1}{\partial t} \quad , \\
h_i(t) &= \frac{\partial b_i}{\partial t} - b_i\frac{\partial c_1}{\partial t} \quad , \\
p_i(t) &= \frac{\partial f_i}{\partial t} - 2b_i\frac{\partial c_2}{\partial t} - \left(4c_2 b_i - f_i\right)\frac{\partial c_1}{\partial t} \quad , \qquad (33) \\
q_1(t) &= \frac{\partial c_1}{\partial t} \quad , \\
q_2(t) &= \frac{\partial c_2}{\partial t} + 2c_2\frac{\partial c_1}{\partial t} \quad . \qquad (34)
\end{aligned}
$$

Substituting Eq.(29), Eq.(32) and Eq.(33) into Eq.(26), and comparing the two sides, we find after simplification

$$
\begin{aligned}
c_1(t) &= \int_0^t dt' \; B_1(t') \quad , \\
c_2(t) &= \exp\left[-2c_1(t)\right] \int_0^t dt' \; B_2(t') \exp\left[2c_1(t')\right] \quad , \\
b_i(t) &= \exp\left[c_1(t)\right] \int_0^t dt' \; E_i(t') \exp\left[-c_1(t')\right] \quad , \\
f_i(t) &= \exp\left[-c_1(t)\right] \int_0^t dt' \; \{F_i(t') + 2B_2(t')b_i(t')\} \exp\left[c_1(t')\right] \quad , \\
a_{ij}(t) &= \int_0^t dt' \; \{A_{ij}(t') + \left[F_i(t') + B_2(t')b_i(t')\right] b_j(t')\} \quad . \qquad (35)
\end{aligned}
$$

Once the coefficients $a_{ij}(t)$, $c_i(t)$, $b_i(t)$ and $f_i(t)$ are known, the operator $U_0(t,0)$ is uniquely determined.

Next, using the above explicit form of the operator $U_0(t,0)$, we can obtain the exact form of the operator $H_I(t)$:

$$
\begin{aligned}
H_I(t) &= \sum_{i=1}^{n} \left\{ f_i(t) + \kappa(t)\theta(t)b_i(t) - \left[d_i(t) + \frac{1}{2}\sigma_i(t)^2 \right] \right\} \frac{\partial}{\partial x_i} + \\
&\quad \{\kappa(t)\theta(t) + 2c_2(t)\} \; \exp\left[c_1(t)\right] \frac{\partial}{\partial r} - r \exp\left[-c_1(t)\right] \quad . \qquad (36)
\end{aligned}
$$

It is easy to see that the operator $U_I(t,0)$ can be expressed in the form

$$
U_I(t,0) = \exp\left[\sum_{i=1}^{n} \xi_i(t) \frac{\partial}{\partial x_i} \right] \mathcal{U}(t,0) \qquad (37)
$$

where

$$
\xi_i(t) = \int_0^t dt' \left\{ f_i(t') + \kappa(t')\theta(t')b_i(t') - \left[d_i(t') + \frac{1}{2}\sigma_i(t')^2 \right] \right\} \qquad (38)
$$

and $\mathcal{U}(t,0)$ satisfies the evolution equation

$$
\mathcal{H}(t)\mathcal{U}(t,0) \equiv \sum_{i=1}^{3} \eta_i(t)\hat{e}_i \mathcal{U}(t,0) = \frac{\partial}{\partial t}\mathcal{U}(t,0) \qquad , \qquad \mathcal{U}(0,0) = 1 \qquad (39)
$$

with

$$
\begin{aligned}
\eta_1(t) &= \left[\kappa(t)\theta(t) + 2c_2(t)\right] \exp\left[c_1(t)\right] \quad , \quad \eta_2(t) = -\exp\left[-\,c_1(t)\right] \quad , \quad \eta_3(t) = 0 \quad , \\
\hat{e}_1 &= \frac{\partial}{\partial r} \qquad , \qquad \hat{e}_2 = r \qquad , \qquad \hat{e}_3 = 1 \quad . \qquad (40)
\end{aligned}
$$

The operators $\hat{e}_i$ form the Heisenberg-Weyl Lie algebra

$$[\hat{e}_1\ ,\ \hat{e}_2] = \hat{e}_3 \qquad , \qquad [\hat{e}_1\ ,\ \hat{e}_3] \ = \ [\hat{e}_2\ ,\ \hat{e}_3] \ = \ 0 \quad . \tag{41}$$

Following a similar procedure as shown above, the operator $\mathcal{U}(t,0)$ is found to be

$$\mathcal{U}(t,0) = \exp\left[\mu_2(t)\hat{e}_2\right]\ \exp\left[\mu_1(t)\hat{e}_1\right]\ \exp\left[\mu_3(t)\hat{e}_3\right] \tag{42}$$

with

$$\mu_1(t) = \int_0^t dt'\ \eta_1(t') \quad , \quad \mu_2(t) = \int_0^t dt'\ \eta_2(t') \quad , \quad \mu_3(t) = \int_0^t dt'\ \mu_2(t')\,\eta_1(t') \ . \tag{43}$$

As a result, we have obtained the exact form of the desired time evolution operator $U(t,0)$ of the pricing equation in Eq.(24). It is then straightforward to show that $P(x_1, x_2, ..., x_n, r, t)$ is given by

$$\begin{aligned} P(x_1, x_2, ..., x_n, r, t) \ &= \ \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \cdots \int_{-\infty}^{\infty} dy_n\ P(y_1, y_2, ..., y_n, r, 0) \\ &\quad \times K(x_1, x_2, ..., x_n, t; y_1, y_2, ..., y_n, 0; r) \end{aligned} \tag{44}$$

where

$$\begin{aligned} & K(x_1, x_2, ..., x_n, t; y_1, y_2, ..., y_n, 0; r) \\ = \ & \frac{1}{\sqrt{(4\pi)^n \mathrm{det}(\mathbf{a})}}\ \exp\left\{\mu_3(t) + c_2(t)\mu_2(t)^2 \exp\left[2c_1(t)\right] + \mu_2(t)\exp\left[c_1(t)\right] r\right\} \times \\ & \exp\left\{-\frac{1}{4}\sum_{i,j=1}^{n}(x_i - y_i + \upsilon_i)(\mathbf{a^{-1}})_{ij}(x_j - y_j + \nu_j)\right\} \end{aligned} \tag{45}$$

is the propagator of the pricing equation in Eq.(24), and

$$\nu_i(t) = b_i(t)r + \xi_i(t) + \mu_2(t) f_i(t)\exp\left[c_1(t)\right] \quad . \tag{46}$$

The matrix $\mathbf{a}(t)$ is the $n \times n$ matrix whose elements are given by $a_{ij}(t)$, and $\mathbf{a^{-1}}(t)$ is its inverse. Furthermore, in terms of the riskless bond function $Q(r,t)$ of the Vasicek model with explicitly time-dependent parameters, we can easily rewrite the propagator $K(x_1, x_2, ..., x_n, t; y_1, y_2, ..., y_n, 0; r)$ and $\nu_i(t)$ as follows:

$$\begin{aligned} & K(x_1, x_2, ..., x_n, t; y_1, y_2, ..., y_n, 0; r) \\ = \ & \frac{Q(r,t)}{\sqrt{(4\pi)^n \mathrm{det}(\mathbf{a})}}\ \exp\left\{-\frac{1}{4}\sum_{i,j=1}^{n}(x_i - y_i + \upsilon_i)(\mathbf{a^{-1}})_{ij}(x_j - y_j + \nu_j)\right\} \end{aligned} \tag{47}$$

and $\nu_i(t) = -\ln\left[Q(r,t)\right] - a_{ii}(t) - \int_0^t dt'\ d_i(t')$.

For illustration, we consider the evaluation of a European call option on the maximum of two assets S_1 and S_2, with a strike price of K. The payoff at expiry for such an option is: $\max(\max(\mathrm{S_1}, \mathrm{S_2}) - \mathrm{K}, 0)$. Then the option price $P(S_1, S_2, r, t)$ is given by

$$P(S_1, S_2, r, t) = I_1 + I_2 + I_3 - K\ Q(r,t) \tag{48}$$

where

$$\begin{aligned}
I_1 &= S_1\ N_2\left(\theta_1, \phi_1, \rho_1\right)\ \frac{\exp\left(-\int_0^t dt'\ d_1(t')\right)}{\sqrt{1+\chi_1^2}} \\
I_2 &= S_2\ N_2\left(\theta_2, \phi_2, \rho_2\right)\ \frac{\exp\left(-\int_0^t dt'\ d_2(t')\right)}{\sqrt{1+\chi_2^2}} \\
I_3 &= K\ Q(r,t)\ N_2\left(\theta_3, \phi_3, \rho_3\right) \\
\chi_1 &= \frac{a_{11}-a_{12}}{\sqrt{\det(\mathbf{a})}} \quad , \quad \chi_2 = \frac{a_{22}-a_{12}}{\sqrt{\det(\mathbf{a})}} \\
\rho_1 &= \frac{\chi_1}{\sqrt{1+\chi_1^2}} \quad , \quad \rho_2 = \frac{\chi_2}{\sqrt{1+\chi_2^2}} \quad , \quad \rho_3 = \frac{a_{12}}{\sqrt{a_{11}a_{22}}} \\
\theta_1 &= -\sqrt{\frac{(1-\rho_1^2)(a_{11}-2a_{12}+a_{22})}{2\cdot\det(\mathbf{a})}}\cdot\left\{\ln\left(\frac{KQ}{S_1}\right) - a_{11} + \int_0^t dt'\ d_1(t')\right\} \\
\phi_1 &= \sqrt{\frac{(1-\rho_1^2)a_{11}}{2\cdot\det(\mathbf{a})}}\cdot\left\{\ln\left(\frac{S_1}{S_2}\right) + a_{11} - 2a_{12} + a_{22} - \int_0^t dt'\ d_1(t') + \int_0^t dt'\ d_2(t')\right\} \\
\theta_2 &= -\sqrt{\frac{(1-\rho_2^2)(a_{11}-2a_{12}+a_{22})}{2\cdot\det(\mathbf{a})}}\cdot\left\{\ln\left(\frac{KQ}{S_2}\right) - a_{22} + \int_0^t dt'\ d_2(t')\right\} \\
\phi_2 &= \sqrt{\frac{(1-\rho_2^2)a_{22}}{2\cdot\det(\mathbf{a})}}\cdot\left\{\ln\left(\frac{S_2}{S_1}\right) + a_{11} - 2a_{12} + a_{22} - \int_0^t dt'\ d_2(t') + \int_0^t dt'\ d_1(t')\right\} \\
\theta_3 &= \frac{\ln(KQ/S_1) + a_{11} + \int_0^t dt'\ d_1(t')}{\sqrt{2a_{11}}} \\
\phi_3 &= \frac{\ln(KQ/S_2) + a_{22} + \int_0^t dt'\ d_2(t')}{\sqrt{2a_{22}}} \quad .
\end{aligned} \tag{49}$$

Here $N_2(\theta, \phi, \rho)$ stands for the bivariate cumulative normal density function. It should be noted that by setting $\sigma_r(t) = \rho_{1r}(t) = \rho_{2r}(t) = \kappa(t) = 0$ in the above price function, we shall obtain the option price $P(S_1, S_2, t)$ for the special case with non-stochastic short-term interest rate. Furthermore, as far as we know, the results in Eq.(48) and Eq.(49) are completely new.

5. Conclusion

In this paper we present a Lie-algebraic technique for the valuation of multi-asset financial derivatives with time-dependent parameters. By exploiting the dynamical symmetry of the pricing partial differential equations of the financial derivatives, the new method enables us to derive analytical closed-form pricing formulae very straightforwardly. We believe that this new approach will provide an efficient and easy-to-use method for the valuation of financial derivatives. Furthermore, this simple Lie-algebraic approach can be easily extended to other financial derivatives with well-defined algebraic structure.

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