Benchmarking Model of Default Probabilities of Listed Companies

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The Basel Committee on Banking Supervision is responsible for proposing capital requirements for internationally active banks. The committee first proposed the Basel New Capital Accord, also known as Basel II, in 1999, with the final version (Basel Committee on Banking Supervision [2004] in June 2004. By year-end 2006, Basel II is expected to replace the original Basel Accord, which was implemented in 1988.

Basel II allows banks to choose among several approaches to determine their capital requirements to cover credit risk. The standardized approach allows less sophisticated banks to use external credit ratings to classify their assets into different risk classes. Over time, banks are expected to evolve to the internal ratings–based (IRB) approaches (foundation and advanced), which rely on their own experience in determining the risk characteristics of various asset classes according to their internal rating systems. For example, the foundation IRB approach for corporate, sovereign, and bank exposures allows banks to provide estimates of probability of default (PD) but requires banks to use supervisory estimates of loss given default (LGD), exposure at default (EAD), and maturity. The advanced IRB approach for such exposures allows banks to provide estimates of all these risk characteristics.

As credit risk measures are estimated by banks, systematic underestimation of such measures and the corresponding regulatory capital in a bank (or a number of banks) will increase the bank's vulnerability to adverse changes in market conditions, in particular during a financial or banking crisis. The safety and stability of the banking system would thus be affected by whether credit risk measures are estimated in a sound and prudent manner. Therefore, the validation methodologies of IRB systems have emerged as one of the important issues of the implementation of Basel II. Validation comprises an assessment of the validity of the risk components EAD, PD, and LGD, and the underlying rating system itself.

For the validation of PDs, there are in general two stages: validation of the discriminatory power of a rating system and validation of the accuracy of the PD quantification. Compared with the evaluation of the discriminatory power, methods for validating the accuracy of the PD quantification are at a much earlier stage. Although one of the methods is backtesting, a major obstacle to backtesting of PDs is the scarcity of data, caused by the infrequency of default events and the impact of default correlation.[1] Even if the five-year requirement of Basel II for the length of time series for PDs is met, the explanatory power of statistical tests will still be limited. Statistical tests alone will be insufficient to establish supervisory acceptance of an internal rating system. Nevertheless, banks should be expected to use various quantitative validation techniques to detect weaknesses in a rating system.

Owing to the limitations of using statistical tests to verify the accuracy of the PD quantification, benchmarking can be a complementary tool for the validation and calibration of PD estimates. Benchmarking involves the comparison of a bank's PD estimates with results from alternative sources. It is quite flexible in the sense that it gives banks and supervisors latitude to select an appropriate benchmark. An important technical issue is the design of the mapping from an individual bank's estimates to the benchmark. Benchmarking seems to be promising and would allow supervisors to make inferences about the characteristics of an internal rating system. It also appears to be part of the whole process of producing internally generated estimates in banks' IRB systems. For example, banks frequently use external and independent references to calibrate their own IRB systems in terms of PDs. Benchmarking internal PD estimates with external and independent PD estimates is implicitly given a special credibility, and deviations from this benchmark provide a reason to review the internal estimates.

This article proposes a benchmarking model for the purpose of IRB validation of listed companies, which is developed using a credit risk model and a simple mapping process. The credit risk model is based on the recent studies of the predictive capability of structural models. Black and Scholes [1973] and Merton [1974] have been the pioneers in the development of the structural models for credit risk of corporates using a contingent-claims framework.[2] They treat default risk as equivalent to a European put option on a firm's asset value and the firm's liability is the option strike. To extend the Merton model, structural models with more complex and dynamic liability structures have been considered by Black and Cox [1976], Longstaff and Schwartz [1995], Briys and de Varenne [1997], Collin-Dufresne and Goldstein [2001], and Hui et al. [2003].[3]

Delianedis and Geske [1999] show that PDs produced by the simple structural models of Merton [1974] and Geske [1977] possess significant and very early information about credit rating migrations. Although the sample of companies that actually default is small, changes in the shape of the term structures of PDs appear to detect impending migrations to default. Leland [2002] finds that PDs generated from the Longstaff and Schwartz model fit actual default rates provided by Moody's Investors Service [1998] for longer time horizons quite well for reasonable parameters with proper calibrations.[4] However, the default boundary in the Longstaff and Schwartz model should be specified as a certain fraction of the principal bond value. This specification imposes a constraint on some of the calibrations for the model (e.g., the asset volatility) that may not be empirically reasonable in order to obtain consistent results.

Hui et al. [2005] propose a structural model where the underlying stochastic variable is the leverage ratio of a firm, which is mean-reverting to a time-dependent target leverage ratio. They show that unlike the Merton model and other variants mentioned earlier, the model is capable of generating term structures of PDs that are consistent with the term structures of actual default rates of credit ratings of BBB and below reported by Standard & Poor's (S&P's) [2002], in particular at longer time horizons. In a special case of the structural model where the liability is assumed to be not mean-reverting, the model converges into a simplified model in which the two main input parameters are the leverage ratio of a firm and its associated volatility. These input parameters can be obtained from market data. The calibrations of the time-dependence and levels of target leverage ratios are not necessary for the simplified model. The use of this simplified model for benchmarking purposes can thus avoid the calibration problem found in Leland [2002]. The following section demonstrates that model PDs generated from this simplified structural model based on market data are consistent with the actual default rates of credit ratings of BBB and below. In view of the capability of the structural model for capturing actual default rates without any specific calibration, credit risk measures of listed companies (with market information about their leverage ratios and associated volatilities) can be obtained from the structural model.

Regarding the mapping process of the benchmarking model, the idea is to associate a company with an external credit rating by mapping the term structure of PDs of the company generated by the structural model to the term structures of default rates reported by a rating agency (e.g., S&P's). According to the actual default rates reported by credit rating agents such as S&P's, different ratings give different term structures of default rates in terms of values and shapes. Such term structures reflect the characteristics of default risk of companies with different ratings.

The term structures of PDs and default rates could be interesting for examining the changing credit structure of individual companies, industries, or the whole economy. The term structure of PDs could contain information about the business cycle. The use of the entire term structure (i.e., up to the cumulative default rate of 15 years) for mapping purposes could also avoid the issue of choosing which particular time horizon (say 1 year or 5 years) of the default rates is the appropriate basis. After mapping the company to an external credit rating (say, S&P's BBB), the corresponding 1-year default rate of the BBB rating is assigned as the benchmark 1-year PD of the company. Such a benchmark PD can be considered as the average 1-year PD estimate based on a pool of companies that have been covered by S&P's rating assessment. The use of such a mapping method could avoid the problem of downward-biased PDs at short maturities produced directly by credit risk models. Although credit ratings provided by rating agencies may not reflect companies' credit quality in a timely way (Delianedis and Geske [1999]), the default rates of the reference data in those rating agencies provide measures of default risk according to the agencies' rating assessment. The mapping process can be thought of as characterizing each company as if it were part of the reference data.

The remainder of the article is organized as follows. In the following section we present the structural model of Hui et al. [2005] used for the benchmarking model. In the section following, we illustrate how the benchmarking model is developed from the structural model and the mapping process, and then the empirical results of the benchmarking model based on data from the United States are presented in the next section. The final section summarizes the findings.

STRUCTURAL MODEL OF TERM STRUCTURES OF PDs

The structural model employed for generating term structures of PDs follows the model proposed by Hui et al. [2005]. In the original model, a firm's liability is assumed to be governed by a time-dependent mean-reverting stochastic process, whereas the firm value (which is defined as market-value capitalization) follows a simple lognormal process. To simplify the model specification, it is assumed that the dynamic process of the liability is not mean-reverting. Therefore the calibration of the time-dependent mean-reverting process is unnecessary in this context. The liability in fact plays no direct role in the simplified model. The key feature is the risk-adjusted probability of the leverage ratio, which is defined as the ratio of a firm's liability to its market-value capitalization, hitting a certain value. A firm's leverage ratio and the risk-free interest rate are the stochastic variables in the model. The leverage ratio is assumed to follow a lognormal diffusion process and the dynamics of the interest rate is drawn from the term structure model of Vasicek [1977], that is, the Ornstein-Uhlenbeck process. The risk-adjusted dynamic of the leverage ratio L is modeled by the following stochastic differential equation:

$$dL = \alpha(t) L dt + \sigma_L(t) L dz_L \tag{1}$$

where $\alpha(t)$ and $\sigma_L(t)$ are the drift and the volatility of L respectively and are time dependent. The drift $\alpha(t)$ is effectively taken as zero in this article.[5] The continuous stochastic movement of the interest rate r follows

$$dr = \kappa(t)\left[\theta(t) - r\right]dt + \sigma_r(t) dZ_r \tag{2}$$

where $\sigma_r(t)$ is the instantaneous volatility. The short-term interest rate r is mean-reverting to long-run mean $\theta(t)$ at speed $\kappa(t)$. The Wiener processes dZ_L and dZ_r are correlated with $dZ_L dZ_r = \rho(t)dt$.

Applying Ito's lemma, the partial differential equation governing the price $P(L,r,t)$ of a corporate discount bond with time-to-maturity of t based on the model is

$$\begin{aligned}\frac{\partial P}{\partial t} = {} & \frac{1}{2}\sigma_L^2(t) L^2 \frac{\partial^2 P}{\partial L^2} + \frac{1}{2}\sigma_r^2(t)\frac{\partial^2 P}{\partial r^2} + \rho(t)\sigma_L(t)\sigma_r(t) L \frac{\partial^2 P}{\partial L \partial r} + \\ & \alpha(t) L \frac{\partial P}{\partial L} + \kappa(t)\left[\theta(t) - r\right]\frac{\partial P}{\partial r} - rP\end{aligned} \tag{3}$$

The bond value is obtained by solving Equation (3) subject to the final payoff condition and the boundary condition. When the firm's leverage ratio is above a predefined level L_0, bankruptcy occurs before bond maturity at $t = 0$. This is consistent with the event of bankruptcy being associated with a high level of the leverage ratio. On the other hand, if the leverage ratio has never breached the predefined level L_0, the payoff to bondholders at bond maturity is the face value of the bond.

As shown in the appendix of Hui et al. [2005], the corresponding default probability, $P_{def}(L, t)$, of a corporate discount bond over a period of time t based on Equation (3) can be approximated by

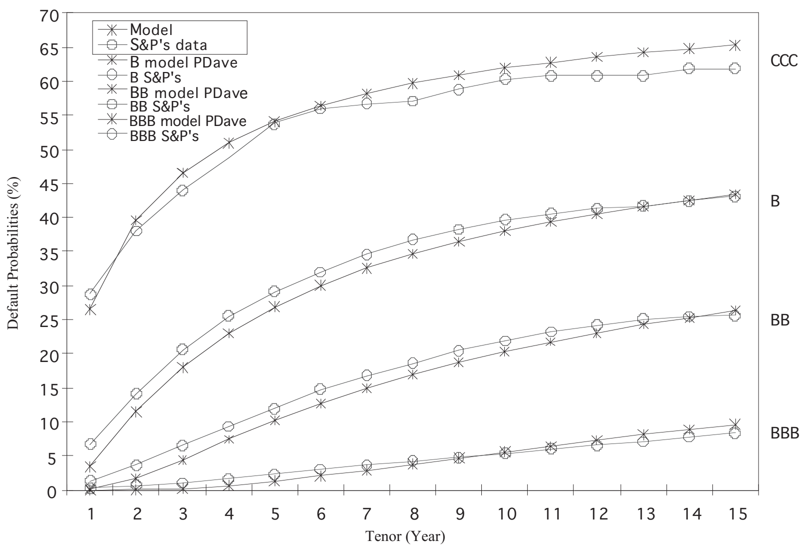
$$\begin{aligned}P_{def}(L,t) = 1 - \Bigg\{ & N\left[\frac{\ln\left(\frac{L_0}{L}\right) - b_2(t)}{\sqrt{2b_1(t)}}\right] - \exp\left[4\beta\left(\ln\left(\frac{L}{L_0}\right) + b_2(t)\right) + 16\beta^2 b_1(t)\right] \times \\ & N\left[\frac{\ln\left(\frac{L}{L_0}\right) + b_2(t) + 8\beta b_1(t)}{\sqrt{2b_1(t)}}\right]\Bigg\}\end{aligned} \tag{4}$$

where $N(.)$ is the cumulative normal distribution func-

EXHIBIT 1

PD Term Structures Generated from the Structural Model and Actual Cumulative Default Rates Reported by S&P's [2002] of Ratings of CCC, B, BB, and BBB

The leverage ratios of ratings CCC, B, BB, and BBB are 0.732, 0.538, 0.495, and 0.315, respectively. The leverage volatilities σ_L of ratings CCC, B, BB, and BBB are 0.299, 0.27, 0.241, and 0.213, respectively.



tion, β is a real number parameter, and $b_1(t)$ and $b_2(t)$ are defined as follows:

$$b_1(t) = \frac{1}{2}\int_0^t \sigma_L^2(t')dt'$$

$$b_2(t) = \int_0^t \gamma(t')dt'$$

$$\gamma(t) = \alpha(t) + \rho(t)\sigma_L(t)\sigma_r(t)a_2(t)\exp[a_1(t)] - \frac{1}{2}\sigma_L^2(t)$$

$$a_1(t) = -\int_0^t \kappa(t')dt'$$

$$a_2(t) = -\int_0^t \exp[-a_1(t')]dt'$$

The parameter β is adjusted such that the approximate solution in Equation (4) provides the best approximation to the exact results by using a simple method developed by Lo et al. [2003] for solving barrier option values with time-dependent model parameters.

The computed PDs within a period of 15 years based on Equation (4) for companies with ratings CCC, B, BB, and BBB are presented in Exhibit 1. The leverage ratios used for individual ratings are based on the industry median reported by S&P's [2001]. The values of σ_L fall close to the asset volatilities of firms with individual ratings estimated by Delianedis and Geske [1999].[6] Other common parameters used in calculations are L_0 = 1.0, α = 0, κ = 1, θ = 5%, σ_r = 0.03162, and ρ = 0. The model term structures of PDs are compared with the term structures of cumulative default rates of the corresponding ratings based on 9,769 companies' assigned long-term ratings from 1981 to 2001 reported by S&P's [2002]. The results in Exhibit 1 show that the model gives the basic shapes and values of the term structures of PDs for ratings of BBB and below, which broadly match with the actual default rates. However, Hui et al. [2005] report that the PDs generated by the model for credit ratings A and above are all much lower than S&P's default rates.[7] This means that the structural model is not capable of differentiating default risks among credit ratings of A– and above. In the benchmarking process, ratings of A– and above are treated as one single rating whose term structure of PDs is the average of those of ratings of A– and above.

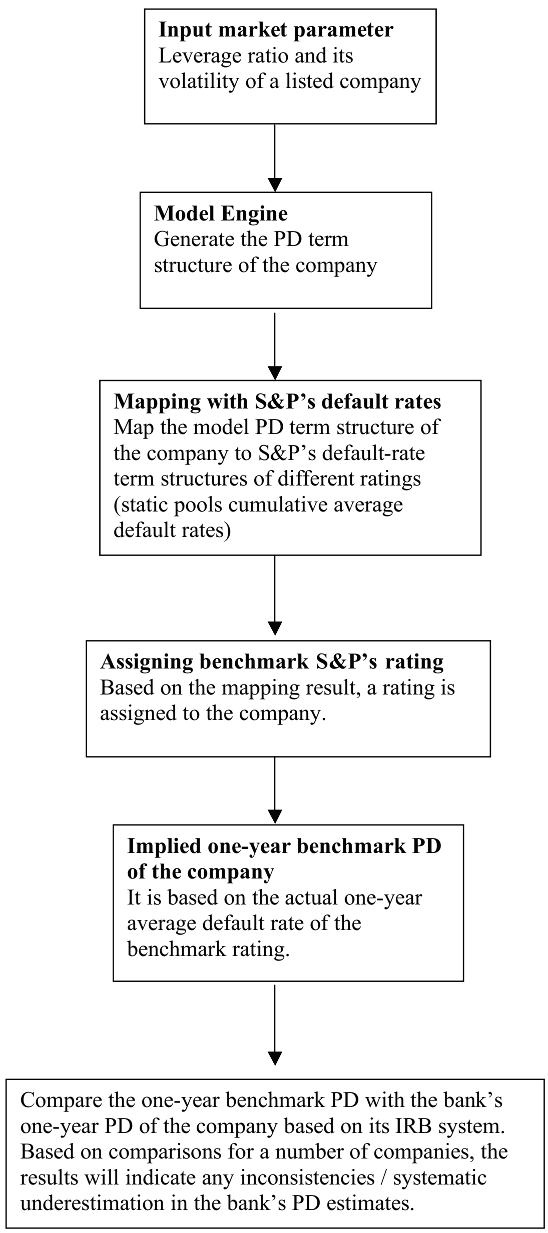
BENCHMARKING PROCESS

According to the requirements under Basel II, PD estimates should be a long-run average of one-year default rates for borrowers. In the benchmarking model, instead of using the structural model presented in the previous section to obtain the one-year PD for benchmarking purposes directly, a corresponding rating of a listed company is obtained by mapping its model term structure of PDs generated by the structural model to the term structure of actual default rates of an external credit rating. The one-year PD of the listed company is thus assigned as the actual one-year average default rates of the corresponding credit rating. According to S&P's [2002], the one-year default rate of a rating is the average default rate over 1981-2001. Such a mapping process can therefore satisfy the requirement of a long-run average.

The benchmarking process involves the structural model and mapping. A mapping is a process of establishing a correspondence between the risk assessment or

EXHIBIT 2

Benchmarking Process of Benchmark PD Estimation



measurement of a company and reference data from external sources such as rating agencies. The process can be thought of as characterizing each company as if it were part of the reference data. The characterizing factor used in this article is the term structures of PDs determined by the structural model and the term structures of cumulative default rates of different ratings reported by S&P's [2002]. Each company is mapped to the reference data based on this characterizing factor.

Exhibit 2 illustrates the benchmarking process. Based on the structural model described earlier, a company's leverage ratio and its volatility are the input parameters used to generate the model term structure of PDs of the company. Using the least square fit, the model term structure of PDs is mapped to the "closest" term structure of default rates of an S&P's rating. Such a rating is assigned to the company as a benchmark rating ranging from CCC to A– and above. The actual one-year average default rate of the benchmark rating gives the corresponding one-year benchmark PD of the company. This benchmarking process can avoid the problem of downward-biased PDs at short maturities, which is common to many credit risk models and in particular contingent-claims models that assume continuous dynamics.

The one-year benchmark PD could be compared with a bank's one-year PD estimate of a company according to its IRB system. The comparison may show whether the PD estimated by the bank is higher or lower than the benchmark PD. Based on comparisons for a number of companies, the results would indicate any inconsistencies or systematic underestimation in the bank's PD estimates relative to the benchmark PDs. Another important application of the benchmarking model is as a means to rank credits on a relative basis. Such benchmark ranking could be compared with the ranking of credits conducted by a bank's internal rating system.

DATA AND EMPIRICAL RESULTS

Leverage Ratios and Their Volatilities

The performance of the benchmarking model in ranking credits is studied in this section. The data used for the analysis consist of 3,943 samples from 193 listed industrial companies in the United States with S&P's ratings from March 1990 to July 2004. Exhibit 3 presents the numbers of companies in the respective S&P's ratings (from CCC to A– and above), which are assigned ordinal numbers. The two input variables (i.e., the leverage ratio and its volatility) of each sample company in the benchmarking model are computed based on the data of its consolidated financial statements and stock prices.

A sample company's liability D includes the principal value of all financial debts, short-term and long-term borrowings, and convertible bonds that participate in the financial leverage of the company. It also includes quasi-financial debts such as capital leases, under-funded pension liabilities, and preferred shares. Non-financial lia-

EXHIBIT 3

Assignment of Ordinal Numbers to S&P's Ratings and Numbers of Sample Companies with S&P's Ratings

S&P's ratings	Ordinal numbers	Numbers of sample companies
A- and above	1	942
BBB+	2	422
BBB	3	618
BBB-	4	430
BB+	5	338
BB	6	368
BB-	7	457
B+	8	207
B	9	91
B-	10	46
CCC	11	24

Source: Bloomberg.

EXHIBIT 4

Mismatch Statistics of Benchmark Ratings versus S&P's Ratings of 3,943 Sample Companies

Mismatch statistics		
Differences	Sample companies %	Cumulative figures
0	21.7%	21.7%
±1	25.9%	47.6%
±2	21.2%	68.8%
±3	12.8%	81.7%
±4	8.8%	90.5%
±5	5.7%	96.1%
±6	2.2%	98.4%

bilities such as accounts payable, deferred taxes, and reserves are not included. The details of the calculation of the company's liability are given in Appendix A. As the company's market-value capitalization can be obtained from its equity values, its leverage ratio can be obtained as the ratio of the liability to market-value capitalization.

In order to obtain the term structures of PDs specified in Equation (4), it is necessary to link the leverage volatility σ_L to the equity volatility σ_S. The values of σ_L are assumed to fall close to the asset volatilities of companies. This means that the volatility of a company's liability is assumed to be immaterial. The daily standard deviation of equity returns $\sigma_S^{(t)}$ is calculated based on a window of 1,000 days, where t is the observation date. The estimate of the daily asset volatility $\sigma_L^{(t)}$ at time t is obtained by applying a gearing ratio to $\sigma_S^{(t)}$ as:

$$\sigma_L^{(t)} = \sigma_S^{(t)} \frac{S}{S+D} \tag{5}$$

where S denotes the company's equity price at time t. The gearing ratio is derived in Appendix B. The annualized leverage volatility $\sigma_L^{(t)}$ at time t is constructed as the square root of 250 times the corresponding daily leverage volatility.

Relative Credit Risk Assessment

For each sample company, we compare its ratings derived from 1) the benchmarking process described earlier (referred to as a benchmark rating) and 2) the S&P's ratings (referred to as a market rating). Ordinal numbers are assigned to individual S&P's ratings as in Exhibit 3. The differences between the market and benchmark ratings based on the ordinal numbers of the sample companies are presented in Exhibit 4. A positive figure refers to an underestimation of the rating (i.e., overestimation of credit risk) of a company by the benchmarking process (e.g., the market rating is BBB+, whereas the benchmark rating is BBB–). The histograms of the mismatch distribution of the market and model rating are presented in Exhibit 5.

The results presented in Exhibit 4 show that the benchmark ratings of 22% of the sample companies exactly match with the market ratings. For a difference of ±1, the coverage is approximately 26% of the sample companies. As the exact matching and the differences of ±1 broadly cover one major rating (e.g., BBB–, BBB, and BBB+ are the subratings of the major BBB rating), 48% of the sample companies show a match between the benchmark ratings and the market ratings in terms of major grades. The results show that the benchmark ratings could broadly track the market ratings. Exhibit 5 shows that most of the sample companies have differences between –2 and +2, though there are some large discrepancies. Such differences mean deviations of two subgrades or less in the S&P's rating scale. According to the matching statistics presented in Exhibit 4, differences between ±2 cover approximately 69% of the sample companies. The most significant outliers tend to be the cases where the benchmarking model imputes a high PD to a company that is rated much better by S&P's. Exhibit 5 also shows that the distribution of mismatches is negatively skewed. This reflects the fact that the benchmarking model tends to assign lower ratings to companies relative to the market ratings assigned by S&P's. This observation could be explained by the finding by Delianedis and Geske [1999] that both rating migrations (mostly rating downgrades) and defaults are detected by the structural models (i.e., the Merton and Geske models) months before the

EXHIBIT 5

Mismatch Distribution of Benchmark Ratings versus S&P's Ratings of 3,943 Sample Companies

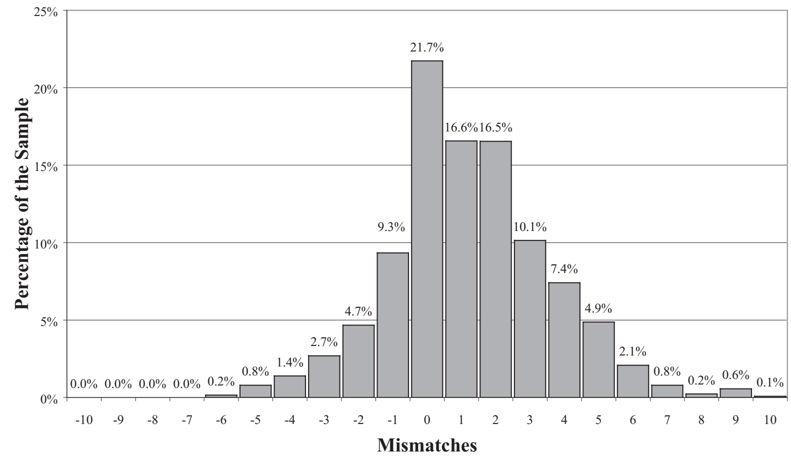
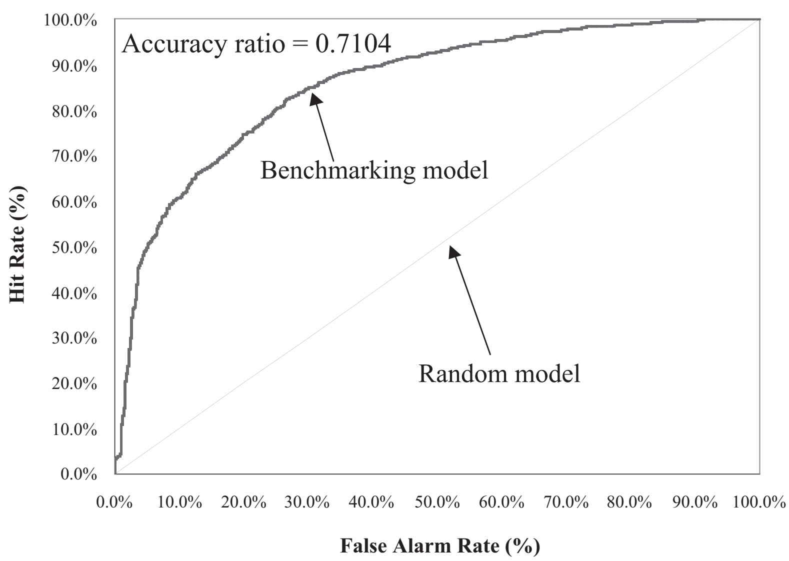


EXHIBIT 6

Receiver Operating Characteristic Curve and Accuracy Ratio of the Benchmarking Model (discriminatory power)



actual events. This means that the structural models would probably give higher credit risk with early information about rating downgrades and default incorporated compared with the actual ratings at a given time. However, the use of different lag times for the matching process does not provide better results for the mismatch statistics. This is because the lag time between early information and actual rating downgrades is not uniform in the sample companies and the lag time also gives "inconsistent" ratings for those sample companies without any risk of rating downgrades and default.

Discriminatory Power

One method to examine the benchmarking model's ability to rank credit risks accurately is through a receiver operating characteristic (ROC), which is a visual tool. The ROC can be constructed as two representative groups of S&P's ratings for investment-rated (i.e., BBB– and above) and non-investment-rated (i.e., BB+ and below) companies among the sample companies. Benchmark ratings are assigned to individual sample companies according to the mapping process discussed earlier. The construction of the ROC does not require the sample composition to reflect the true proportion of investment-rated and non-investment-rated companies. Concavity of the ROC is equivalent to the conditional PDs being a decreasing function of the underlying ratings, and non-concavity indicates suboptimal use of information in the specification of the rating function.

The ROC curve is constructed as follows. From the benchmark PDs it is determined which sample companies will be categorized as investment-rated or non-investment-rated companies. A ranking of the companies is established in line with the assessment of their risk according to their benchmark PDs, starting with the riskiest company and ending with the company classified as being the least risky. Each company is checked to see whether its benchmark PD correctly assigns it as an investment-rated or non-investment-rated company by comparison with its S&P's rating.

The hit rate *HR(PD)* is defined as

$$HR(PD) = \frac{H(PD)}{N_{NI}} \tag{6}$$

where *H(PD)* is the number of companies assigned correctly as non-investment-rated companies based on the benchmarking model and N_{NI} is the total number of non-investment-rated companies in the samples. This means that the hit rate is the fraction of non-investment-rated companies that are classified correctly. The false alarm rate *FAR(PD)* is defined as

$$FAR(PD) = \frac{F(PD)}{N_I} \tag{7}$$

EXHIBIT 7

Degree of Association between Benchmark Ratings and S&P's Ratings of 3,943 Sample Companies

Degree of Association			
Type of statistics	**Estimates**	**Asymptotic standard error**	***p*-value of no association**
Kendall's tau (τ_b)	0.5241	0.0090	0.0000
Stuart's tau (τ_c)	0.5070	0.0088	0.0000
Gamma (Γ)	0.5873	0.0101	0.0000

where *F(PD)* is the number of false alarms, that is, the number of investment-rated companies that are classified incorrectly as non-investment-rated companies based on the benchmarking model. The total number of investment-rated companies in the samples is denoted by N_I. The ROC curve is a plot of *HR(PD)* versus *FAR(PD)*, which is illustrated in Exhibit 6.

A model's performance is the better the steeper the ROC curve is at the left end and the closer the ROC curve's position is to the point (0,1). This means that the model is better, the larger the area under the ROC curve is. The quality of the benchmarking model is measured by the accuracy ratio (*AR*). Engelmann et al. [2003] prove a relation between *AR* and the area under the ROC curve. The *AR* is defined as

$$AR = 2\int_0^1 HR(FAR)\,d(FAR) - 1 \tag{8}$$

The *AR* is 0 for a random model without discriminatory power and it is 1.0 for a perfect model. As the *AR* of the benchmarking model in Exhibit 6 is 0.71, the benchmarking model has adequate discriminatory power for ranking the credit risks of the sample companies and is a reasonable model in practice.

Measures of Association

Measures of the association between the market and benchmark ratings are studied in this subsection. One standard measure is a simple correlation statistic. However, correlation can be overly influenced by outlier data. The correlation statistic is likely to be dominated by the companies with the highest credit risk, and high-credit-quality companies are likely to be given little weight. To address these concerns, rank correlation statistics (Kendall's tau, Stuart's tau, and gamma) are examined. Such rank order statistics measure the degree of co-monotonic dependence of two random variables. The notion of co-monotonic dependence generalizes linear dependence that is expressed via (linear) correlation. In particular, any pair of random variables with correlation 1 (i.e., any linearly dependent pair of random variables) is co-monotonically dependent. In addition, as soon as one of the variables can be expressed as any kind of increasing transformation of the other, the two variables are co-monotonic.

Given a pair of random variables (*X*, *Y*), Kendall's tau is defined as

$$\tau_b = P(X_1 < X_2, Y_1 < Y_2) + P(X_1 > X_2, Y_1 > Y_2) - P(X_1 < X_2, Y_1 > Y_2) - P(X_1 > X_2, Y_1 < Y_2) \tag{9}$$

where (X_1, Y_1) and (X_2, Y_2) are independent copies of (*X*, *Y*). Hence, τ_b can be seen as the difference between two probabilities, namely, the probability that the larger of the two *X*-values is associated with the larger of the two *Y*-values and the probability that the larger *X*-value is associated with the smaller *Y*-value. Two series that are identical will have a statistic of 1, whereas a statistic of 0 indicates no association. Detailed definitions of the rank order statistics (i.e., Kendall's tau, Stuart's tau, and gamma) are given in Appendix C.

The measures of association for the benchmark ratings with the market ratings are presented in Exhibit 7. The calculations of the statistics are based on the method in Brown and Benedetti [1977]. The rank correlation statistics of Kendall's tau, Stuart's tau, and gamma are 0.5241, 0.5070, and 0.5873, respectively. The asymptotic standard errors of the statistics, which are obtained by assuming the null hypothesis of no association between the variables (i.e., $\tau_b = 0$, $\tau_c = 0$, and $\Gamma = 0$), are equal to or less than 1%. The *P*-values of no association of the statistics are zero. The results show that the benchmark ratings could broadly track the market ratings and the association between them is adequately significant. This indicates that the likelihood of the rating agencies (e.g., S&P's) and benchmarking model ranking companies in the same order is reasonably high.

SUMMARY

This article presents a benchmarking model for the purpose of IRB validation of PD estimates of publicly listed companies. In view of the capability of the structural credit risk models for capturing term structures of actual default rates, credit risk measures of listed companies can be obtained from the structural model developed by Hui et al. [2005] without any specific calibration in this article. The model inputs of leverage ratios and associated volatilities are available in market data. The benchmarking model assigns benchmark ratings and one-year PDs to companies by mapping the term structures of PDs of the companies generated by the chosen structural model to the term structures of default rates reported by S&P's. The empirical results show that the benchmark ratings could broadly track the S&P's ratings of the U.S. sample companies. The association between them is statistically significant. The results demonstrate that the benchmarking model has adequate discriminatory power of ranking the credit risk of the sample companies in terms of differentiating investment-rated and non-investment-rated companies. Benchmark PDs obtained from the model could thus be used as alternative external and independent PD estimates for comparison with banks' internal PD estimates of listed companies. Significant deviations from this benchmark provide a reason to review the banks' internal estimates and their credit rating processes.

APPENDIX A

A company's leverage ratio is defined as the ratio of its liability to market-value capitalization. The liability D is determined from financial data in consolidated statements. Using Bloomberg's data, the financial debt of the company is the sum of the short-term and long-term interest-bearing financial obligations (e.g., loans and bonds) and 50% of other non-interest-bearing obligations (see Pan [2001]). The 50% weight is assumed because some of these obligations (e.g., pension liabilities) are similar to the nature of financial liabilities, whereas others of them (e.g., provisions) are not.

The financial data in the consolidated financial statement contain 100% of the financial liabilities of subsidiaries even though the parent company does not fully own the subsidiaries. This may therefore exaggerate the liabilities of the company. To adjust for this, a portion of the liabilities of its subsidiaries that are not owned by the company should be subtracted from the financial debt. The liability of the company is equal to its financial debt less the minority interest that represents the portion of interest that the parent company does not own in the subsidiaries. In the calculation, the amount of the minority interest is limited to no more than half of the financial debt. Therefore, the liability D is

$$D = \text{financial debt} - \text{Min (minority interest, financial debt/2)}$$

APPENDIX B

The derivation of the gearing ratio in Equation (5) follows the methodology used in Pan [2001]. Let S and σ_S denote a company's equity price and its equity volatility respectively. The volatility σ_L of the company is assumed to fall close to the volatility σ_V of the company's asset value. This means that the volatility of the company's liability is assumed to be immaterial. In general, σ_S and σ_L are related through

$$\sigma_S = \sigma_V \frac{V}{S}\frac{\partial S}{\partial V} \tag{A-1}$$

The distance to default measure η is defined as the number of annualized standard deviations separating the company's current equity value from the default threshold such that

$$\eta = \frac{1}{\sigma_V}\log\left(\frac{V}{D}\right) = \frac{V}{\sigma_S S}\frac{\partial S}{\partial V}\log\left(\frac{V}{D}\right) \tag{A-2}$$

where V and D are the company's asset value and liability, respectively. To obtain the gearing ratio, the boundary conditions of Equations (A-1) and (A-2) are examined. The first boundary condition is the behavior of V near D, that is, the default threshold. As default approaches, S approaches zero. Thus,

$$V\big|_{S=0} = D \tag{A-3}$$

at the boundary and

$$V \approx D + \frac{\partial V}{\partial S} S \tag{A-4}$$

near the boundary. By substituting Equation (A-4) into Equation (A-2) we have

$$\eta \approx 1/\sigma_S \tag{A-5}$$

near the boundary. The second boundary condition is far from the default barrier (i.e., $S >> D$). Here, we have

$$S/V \to 1 \tag{A-6}$$

This leads to an approximation for η under Equation (A-2) as

$$\eta \simeq \frac{1}{\sigma_S}\log\left(\frac{S}{D}\right) \tag{A-7}$$

The simplest expressions for V and η that simultaneously satisfy the near default boundary conditions (A-3) and (A-5) and the far from default conditions (A-6) and (A-7) are $V = S + D$

and

$$\eta = \frac{S+D}{\sigma_S S}\log\left(\frac{S+D}{D}\right) \quad \text{(A-8)}$$

Thus, Equations (A-2) and (A-8) give

$$\sigma_L = \sigma_S \frac{S}{S+D} \quad \text{(A-9)}$$

by assuming $\sigma_L \approx \sigma_V$. Equation (A-9) relates the leverage volatility to the observable equity volatility.

APPENDIX C

In the following illustration, all measures are defined by their sample analogs. Let a_{ij} denote the observed frequency in cell (i, j) in an $I \times J$ contingency table. Let $r_i = \Sigma_j a_{ij}$ be the i-th row total, $c_j = \Sigma_i a_{ij}$ the j-th column total, and $N = \Sigma_i \Sigma_j a_{ij}$ be the total frequency. Let

$$A_{ij} = \sum_{k<i}\sum_{t<j} a_{kt} + \sum_{k>i}\sum_{t>j} a_{kt} \quad \text{(A-10)}$$

$$D_{ij} = \sum_{k>i}\sum_{t<i} a_{kt} + \sum_{k<i}\sum_{t>j} a_{kt} \quad \text{(A-11)}$$

$$P = \sum_i \sum_j a_{ij} A_{ij} \quad \text{(A-12)}$$

$$Q = \sum_i \sum_j a_{ij} D_{ij} \quad \text{(A-13)}$$

A_{ij} is the total frequency of the cells whose indices are either both greater than or both less than (i, j). D_{ij} is the total frequency of the cells that have one index greater and one index less than (i, j). Thus P is twice the number of agreements, and Q is twice the number of disagreements in the ordering of the cell indices when all pairs of observations are compared. In this context, an interpretation of P is the probability that for a randomly chosen pair of companies, both the market and benchmarking model will rank the firms in the same order. Q represents the probability that the market and benchmarking model disagree on the ranking. The definitions of P and Q exclude ties, where ties are defined as pairs of observations sharing at least one common index. As given in Kendall [1955], Kendall's tau τ_b is estimated by

$$\tau_b = (P-Q) \Big/ \left[\left(N^2 - \sum_i r_i^2\right)\left(N^2 - \sum_j c_j^2\right)\right]^{\frac{1}{2}} \quad \text{(A-14)}$$

and Stuart's tau τ_c is estimated by

$$\tau_c = (P-Q)/[N^2(m-1)/m] \quad \text{(A-15)}$$

where $m = \min(I, J)$. Goodman and Kruskal [1954] propose the measure of association gamma estimated by

$$\Gamma = (P-Q)/(P+Q) \quad \text{(A-16)}$$

These three measures have the same numerator but differ in the manner by which they are normalized. It is noted that $|\tau_c| \le |\tau_b| \le |\Gamma$ (see Brown and Benedetti [1977]).

ENDNOTES

The conclusions herein do not represent the views of the Hong Kong Monetary Authority.

[1]Owing to correlation between defaults in a portfolio, observed default rates can systematically exceed the critical PD values if these are determined under the assumption of independence of the default events.

[2]The second approach is the reduced-form models in which time of default is assumed to follow a stochastic process governed by its own distribution, which is characterized by an intensity or hazard rate process. This approach has been considered by Jarrow and Turnbull [1995], Jarrow et al. [1997], Madan and Unal [1998], and Duffie and Singleton [1999]. Their models in general focus on more sophisticated characterization of the hazard process. The derived pricing formulas can be calibrated to market credit spreads. Some extensions explore assumptions surrounding recovery rate, risk-free interest rate processes, and contract boundary conditions.

[3]Also see the survey of Bohn [2000].

[4]The predicted PDs are too low for short maturities. The problem of downward-biased PDs at short maturities is, however, common to all contingent-claims credit risk models that assume continuous dynamics.

[5]If a firm's market-value capitalization and liability are assets that some agent is willing to hold, their risk-adjusted drift will be equal to the instantaneous interest rate. The drift of their ratio L is therefore zero.

[6]The volatility of a company's liability is assumed to be immaterial.

[7]The reason is that the problem of downward-biased default risk of companies with good credit quality is common to models that assume continuous dynamics of the underlying variables.

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