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Valuation Model of Defaultable Bond Values in Emerging Markets

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Abstract. This paper develops a model to value defaultable bonds in emerging markets. Default occurs when some signaling process hits a pre-defined default barrier. The signaling variable is considered to be some macro-economic variables such as foreign exchange rates. The dynamics of the default barrier depend on the volatility and the drift of the signaling variable. We derive a closed-form solution of the defaultable bond price from the model as a function of a signaling variable and a short-term interest rate. The numerical results show that the model values generated by using foreign exchange rates as the signaling variables can broadly track the market credit spreads of defaultable bonds in South Korea and Brazil. Given an expected level of the foreign exchange rate, defaultable bond values under a stressed market situation can be obtained.

Key words: risky bonds, credit risk, emerging markets, stress tests.

1. Introduction

There are generally two approaches to model the valuation of defaultable bonds. The first approach consists of structural models that treat default risk equivalent to a European put option on the corporate asset value and the corporate liability is the option strike. Black and Scholes (1973) and Merton (1974) have been the pioneers in this approach. In Merton's framework, default occurs only at bond maturity when the asset value is less than the liabilities due to the bond, and the firm is insolvent. To cope with the possibility of early default before bond maturity, Black and Cox (1976) assume a bankruptcy-triggering level for the corporate assets whereby default can occur at any time. Longstaff and Schwartz (1995) extend the risky debt model of Black and Cox to allow interest rate to follow the Ornstein–Uhlenbeck process.[1] Default occurs when the corporate asset value is below a constant or deterministic bankruptcy-triggering barrier. Upon bankruptcy triggered by touching the barrier, bondholders receive an exogenously given number of default-free bonds.

The second approach is the reduced-form models in which default time is a stopping time of some given hazard rate process and the payoff upon default is specified exogenously. This approach has been considered by Artzner and Delbaen

(1992), Madan and Unal (1993), Jarrow, Lando, and Turnbull (1994), Jarrow and Turnbull (1995), and Duffie and Singleton (1997). The derived pricing formulas can be calibrated to market credit spreads.

Under stressed market conditions in emerging markets, equity markets and bond markets would be volatile and illiquid. It would be difficult to determine actual assets and liabilities of the bond issuers and to input them as parameters in the structural models. In addition, sometimes it is difficult to collect sufficient market data of credit spreads of bonds in emerging markets due to illiquidity of trading. This causes difficulties in the calibration of the reduced-form models. In developing a defaultable bond valuation model for emerging markets, we therefore employ a middle ground model between the structural models and the reduced-form models to price defaultable bonds. This model was first proposed by Cathcart and El-Jahel (1998). In the model, default occurs when some signaling process hits a lower constant default barrier. The model assumes the signaling process for each firm that determines the occurrence of default rather than the value of the assets of the firm. The signaling process can capture factors that can affect the probability of default. The use of the signaling process is also appropriate for entities such as sovereign issuers that issue defaultable debts but do not have an identifiable collection of assets. When the signaling variable drops below the default barrier, bondholders receive an exogenously specified number of default-free bonds. Regarding the underlying interest rate, it is assumed to follow a mean-reverting square root process that is uncorrelated with the signaling process. An analytical defaultable bond price solution is derived from the model by Cathcart and El-Jahel. The term structures of credit spreads derived from the model match with the empirical evidence.[2] However, since the solution is not in closed form and is expressed as inverse Laplace transforms, numerical techniques need to be employed to perform the transforms. This may impose some numerical difficulties to obtain numerical results.

While Cathcart and El-Jahel's paper does not identify what the signaling variable is, we suggest that the signaling variable could be some macro-economic variables such as foreign exchange rates in the valuation model. Given a default level for a macro-economic variable, values of defaultable bonds can be obtained from the model. In addition, this paper improves the Cathcart and El-Jahel model by incorporating a drifted default barrier that is governed by the volatility and the drift of the signaling variable.[3] The contribution of the signaling variable's dynamics to the barrier's dynamics is adjusted by a free parameter β. When the parameter β is equal to zero, the model is reduced to the case of a constant default barrier proposed by Cathcart and El-Jahel. The model in this paper is therefore characterised by a flexible risk adjustable default barrier. Expected or more realistic default scenarios can be put into the model through adjusting the parameter β. For example, if the future default level is expected to be different from the current one, the default level can be adjusted via the parameter β to an expected future level. The dynamics of the short-term interest rate is assumed to follow the square root process. From the given valuation model framework, we derive a closed-form defaultable bond solution in terms of a cumulative normal distribution function. Therefore, no sophisticated numerical technique is needed to compute the solution. The model parameters such as volatility of the signaling variable, volatility, drift and mean-level of the interest rate are time dependent in the derivation.

During the recent financial crisis in emerging markets, many corporate bonds and sovereign bonds denominated in foreign currencies (e.g., U.S. dollar and Japanese Yen) were under stressed conditions and their prices sank. These events raise a question how to develop a valuation model for these bonds such that the bondholders, for example banks, can assess their capital adequacy under stressed market conditions. The quantitative results from the valuation model could identify plausible stress scenarios to which bondholders could be exposed. By putting these economic factors under stress scenarios in the model, the corresponding bond values could be obtained from the model and used to assess the potential extraordinary losses of bondholders.

The scheme of this paper is as follows. In the following section, we discuss the valuation model framework of discount defaultable bonds with a drifted default barrier, and derive the corresponding closed-form pricing formula with time-dependent parameters. Numerical results of credit spreads calculated from the pricing formula are shown in Section 3. The results are compared with the actual market credit spreads of Korean Government bonds and Brazilian Government bonds. In the last section we shall summarise our investigation.

2. Model Framework

2.1. VARIABLES AND THEIR DYNAMICS

In the valuation of discount defaultable bonds, we assume a continuous-time framework, and let the short-term interest rate and the signaling process be stochastic variables. The assumptions and the dynamics of the variables follow those used in the Cathcart and El-Jahel model. We regard the signal as tradable and assume that the signaling process be risk-adjusted from the beginning. The dynamics of the variables are as follows:

(i) Dynamics of the short-term interest rate.

The risk-adjusted dynamics of the short-term interest rate r are drawn from the term structure model of Cox, Ingersoll, and Ross (CIR) (1985), i.e., the square-root process:

$$dr = \kappa\,(t)\,[\theta\,(t) - r]dt + \sigma_r(t)\sqrt{r}dz_r\,, \tag{1}$$

where the short-term interest rate is mean-reverting to long-run mean $\theta\,(t)$ at speed $\kappa\,(t)$, and the stochastic term has a standard deviation proportional to $\sqrt{r}$. The CIR model rules out negative rates that could happen in an Ornstein–

Uhlenbeck process. Therefore, the square-root process is more appropriate in a low interest rate scenario, such as current Japanese interest rates.

(ii) Dynamics of the signaling variable.

The risk-adjusted dynamic of the signaling variable S are assumed to follow a lognormal diffusion process:

$$dS = \alpha\,(t)\,Sdt + \sigma_S\,(t)\,Sdz_S\,, \tag{2}$$

where $\alpha\,(t)$ and $\sigma_S\,(t)$ are the drift and the volatility of S respectively and are time dependent. The signaling process can capture factors that can affect the probability of default.

(iii) Assumption of no correlation.

The Wiener processes dz_S and dz_r are assumed to be uncorrelated. The no-correlation assumption between the signaling process and the interest rates is in line with most of the reduced-form models, where the hazard rate of default process is assumed to be uncorrelated with the interest rates.

While Cathcart and El-Jahel's paper does not identify the signaling variable, we suggest that the signaling variable could be some macro-economic variables – GDP growth rate, long-term interest rates, and foreign exchange rates. Altman (1990) and Wilson (1997a, b) study the explanatory power of macro-economic factors in predicting the number of defaults. Altman uses first order differences, the explanatory variables being the percentage change in real GNP, percentage change in the money supply, percentage change in the Standard & Poor index and the percentage change in new business formation. Altman finds a negative relation between changes in these variables and changes in the aggregate number of business failures. Wilson examined the effects of macro-economic variables – GDP growth rate, unemployment rate, foreign exchange rates, and long-term interest rates in estimating default rates. All of these studies suggest that credit spreads are affected by common economic underlying influences. In the numerical results section, we use foreign exchange rates as the signaling variables to demonstrate the application of the model to defaultable bond valuation under stressed market conditions.

2.2. MODEL OF DEFAULTABLE DISCOUNT BONDS

We let the price of a discount defaultable bond be $P(S, r, t)$. The partial differential equation governing the bond is[4]

$$\begin{aligned}\frac{\partial P}{\partial t} &= \frac{1}{2}\sigma_S\,(t)^2\,S^2\frac{\partial^2 P}{\partial S^2} + \frac{1}{2}\sigma_r\,(t)^2\,r\frac{\partial^2 P}{\partial r^2} + \\ &\quad + \alpha\,(t)\,S\frac{\partial P}{\partial S} + \kappa\,(t)\,[\theta\,(t) - r]\frac{\partial P}{\partial r} - rP\,.\end{aligned} \tag{3}$$

The value of a defaultable bond is obtained by solving Equation (3) subject to the final payoff condition and the boundary condition imposed by the default barrier. We propose the default barrier $H(t)$ to have drifted dynamics, which are determined by the drift and the volatility of the signaling variable. The default barrier is expressed as:

$$H(t) = S_0 \exp\left[-C(t) - \gamma B(t)\right], \tag{4}$$

where

$$B(t) = \frac{1}{2}\int_0^t \sigma_S(\tau)^2 \, d\tau, \quad C(t) = \frac{1}{2}\int_0^t \left[\alpha(\tau) - \frac{\sigma_S(\tau)^2}{2}\right] d\tau .$$

S_0 is the pre-defined value of the barrier and γ is a real adjustable parameter controlling the movement of the barrier. For a special case of constant α and σ_S, the moving barrier will simply be reduced to the form:

$$H(t) = S_0 \exp\left[-\beta\left(\alpha - \frac{\sigma_S^2}{2}\right) t\right] \tag{5}$$

and β is a real number parameter to adjust the rate of the drift. The movement of the barrier can be interpreted as a mean drift (adjusted by β) arising from the dynamics of S. When the term $\left(\alpha - \sigma_S^2/2\right)$ is less than zero, the barrier level increases with the time to maturity for a positive β. On the other hand, given a negative β, the barrier level decreases with the time to maturity. It means that given an initial S_0 as the pre-defined default level, the probability of default increases with the value β when $\sigma_S^2/2$ is higher than the drift α. Given the same β, the barrier moves away from S_0 with time to maturity at a faster rate when σ_S is higher. This demonstrates the effect of σ_S on early default risk of a defaultable bond. It is noted that when the parameter β is put to be zero, the case of a constant barrier is obtained, i.e., recovering Cathcart and El-Jahel's model. For $\beta = -1$, the barrier basically moves with the mean drift of the signaling variable. The barrier dynamics incorporating the adjustable mean drift of the signaling variable are more flexible than the constant barrier specified in the Cathcart and El-Jahel model.

When S breaches the barrier $H(t)$, bankruptcy occurs before maturity at $t = 0$. The payoffs to bondholders are specified by

$$P(S = H, r, t) = WFQ(r, t) \quad t > 0;\ W < 1, \tag{6}$$

where $Q(r, t)$ is the default-free bond function according to the CIR model and F is the bond face value. On the other hand, if S has never breached the barrier, the payoff to bondholders at the bond maturity is

$$P(S, r, t = 0) = F \quad S > H(t). \tag{7}$$

When $W = 0$, bondholders receive nothing in a financial reorganisation; and when $W > 0$, bondholders receive new securities in exchange for their original claims. Empirical studies on the value of W have been conducted by Altman (1991, 1992).

Altman calculates the average W for a sample of defaulted bonds over the period 1971–1991. He finds average W are 47.2% for debt rated A when issued, 39.5% for debt rated BBB, and 30.7% for debt rated BB.

The solution of Equation (3) subject to the boundary condition and the final payoff condition of Equations (4), (6) and (7) is

$$P(S,r,t) = FQ(r,t)\left\{W + (1-W)\left[N(\xi_1) - N(\xi_2)\exp(\xi_3)\right]\right\}, \tag{8}$$

where

$$\xi_1 = \frac{\ln(S/S_0) + C(t)}{\sqrt{2B(t)}}, \quad \xi_2 = -\xi_1 - \gamma\sqrt{2B(t)},$$

$$\xi_3 = \gamma\left[\ln(S/S_0) + C(t) + \gamma B(t))\right]$$

and N is a cumulative normal distribution function. For a special case of constant α and σ_S, the solution of Equation (3) subject to the barrier movement of Equation (5) is

$$P(S,r,t) = FQ(r,t)\left\{W + (1-W)\left[N(d_1) - \left(\frac{S}{S_0}\right)^{\eta_1}\exp(\eta_2)N(d_2)\right]\right\}, \tag{9}$$

where

$$d_1 = \frac{\ln(S/S_0) + \left(\alpha - \sigma_S^2/2\right)t}{\sigma_S\sqrt{t}}, \quad d_2 = -d_1 - 2(\beta-1)\left(\frac{\alpha - \sigma_S^2/2}{\sigma_S}\right)\sqrt{t},$$

$$\eta_1 = -2(1-\beta)\left(\frac{\alpha - \sigma_S^2/2}{\sigma_S^2}\right), \quad \eta_2 = \eta_1\beta\left(\alpha - \sigma_S^2/2\right)t.$$

The detailed derivation of the solutions in Equations (8) and (9) is given in the appendix. It is easy to show from Equation (8) that the defaultable bond price is equal to the recovery value $WFQ(r,t)$ when S breaches the barrier. The price of the defaultable bond has two components, the recovery value $WFQ(r,t)$ and the risk-adjusted value of no default. Therefore, the probability of default $f(S,t)$ which can be obtained from Equation (8) is

$$f(S,t) = 1 - \left[N(\xi_1) - N(\xi_2)\exp(\xi_3)\right]. \tag{10}$$

The default premium consists of the product of $f(S,t)$ and the present value of the loss in case of default $(1-W)FQ(r,t)$. Therefore, Equation (8) can be rewritten as

$$P(S,r,t) = FQ(r,t) - (1-W)FQ(r,t)f(S,t)\ . \tag{11}$$

By using Equation (8), coupon bonds can be valued as simple portfolios of discount bonds. The same analysis can be applied to Equation (9).

3. Numerical Results and Market Data

The credit spread C_s of a defaultable discount bond price $P(S, r, T)$ based on Equation (9) with time to maturity T and face value F is given as

$$C_s(S, r, T) = -\frac{1}{T} \ln \frac{P(S, r, T)}{FQ(r, T)}. \tag{12}$$

We use credit spreads of two U.S. dollar denominated bonds issued by the South Korean Government (SKG) to compare with the numerical results of the model. The two bonds were issued in April 1998 with maturity of 5 years and 10 years, and semi-annual coupon payment of 8.75% and 8.875% respectively. When the South Korean Won (KRW) severely depreciates against the USD, the ability of the SKG to repay its liabilities in the USD SKG bonds is put under pressure. This relationship was observed during the crisis in December 1997. The crisis led to the country restructuring the USD 24 billion in short-term debts. Therefore, it is reasonable to choose the signaling variable S to be the exchange rate of the KRW against the USD. The daily SKG bonds' market credit spreads[5] over the corresponding U.S. Treasury bonds from April 10, 1998 to February 25, 2000 are compared with model values. To calculate the model values, the initial S_0 of the default barrier is set to be 1982 KRW/USD, which is slightly above the highest value 1962 KRW/USD during December 1997. The drift α is taken to be the short-term interest rate differential between KRW and USD. While the USD interest rate is stochastic in the valuation due to the USD denomination of the bonds, we assume the interest rate differential between KRW and USD to be non-stochastic and set it to be –2%. This assumption of non-stochastic interest rates is commonly used in option pricing of foreign exchanges. It is considered to be adequate to demonstrate the effect of the signaling variable on bond values based on non-stochastic model parameters. The volatility σ_S of the KRW/USD rate used is the market historical volatility which is approximated as decaying with the rate of $\exp(-6\tau)$ from 25% as at April 10, 1998 to 11% as at February 25, 2000, where τ is the time from April 10, 1998. Therefore, different values of constant σ_S are applied to pricing the bonds according to different time to maturity T. Since the two SKG bonds were rated BBB when they were issued, value W is set to be 0.4 that is consistent with the empirical finding in Altman (1991, 1992). The default barrier is assumed to mainly follow the mean drift of S and β is set to be –1. Other parameters used in the calculations are $\sigma_r = 0.078$, $r = 5\%$, $\theta = 9\%$ and $\kappa = 0.5$.

The model values and market credit spreads of the 5-year and 10-year SKG bonds are illustrated in Figures 1 and 2 respectively. The figures also include KRW/USD exchange rates during and after the crisis period. Both figures show that the model values follow the market credit spreads closely after September 1998 when the spreads of both bonds peaked at 9.63% in early September 1998. The surges in the credit spreads were obviously not due to the foreign exchange rate that did not depreciate much. The drop in the bond prices was caused by global concern about the default of Russian Government bonds. On the other hand,

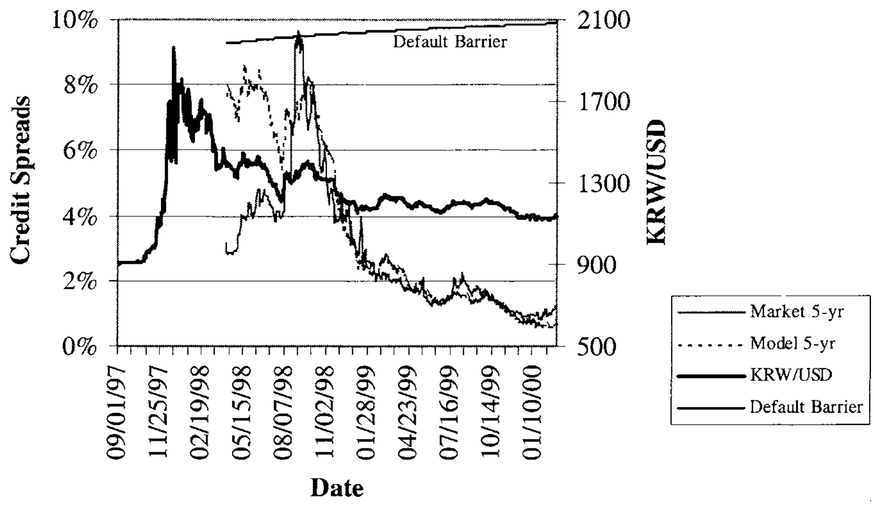

Figure 1. Market and model credit spreads of a 5-year USD denominated South Korean Government bond with $S_0 = 1982$, $\beta = -1$, $\alpha = -2\%$, $W = 0.4$, $\sigma_r = 0.078$, $r = 5\%$, $\theta = 9\%$ and $\kappa = 0.5$.

the model values are higher than the market spreads before the surges in market credit spreads. Since the signaling variable only depends on the foreign exchange rate, other financial factors are not captured in the model and their impact on the credit quality of the bonds could not be reflected from the model values. However, the model values are still highly correlated with the market credit spreads. The correlations of the two pairs of series are both 0.84. If the series are taken from October 1, 1998 onwards, the correlations are 0.97 and 0.96 for the 5-year bond and the 10-year bond respectively.

We use other examples, credit spreads of two USD denominated bonds issued by the Brazilian Government (BLG), to compare with the numerical results of the model. The two bonds were respectively issued in April 1998 with maturity of 10 years and semi-annual coupon payment of 11.625%, and in April 1999 with maturity of 5 years and semi-annual coupon payment of 9.375%. Similar to the South Korean example, when the Brazilian Real (BLR) was allowed to float and then be devalued during the currency crisis in January 1999, the default risk of the BLG USD bond increased. Therefore, the signaling variable S is also chosen to be the exchange rate of the BLR against the USD. The daily BLG bonds' credit spreads over the corresponding US Treasury bonds from January 12, 1999 (for the 10-year bond) and April 22, 1999 (for the 5-year bond) to March 3, 2000 are compared with model values. To calculate the model values, the initial S_0 of the default barrier is set to be 2.4 BLR/USD. The highest BLR/USD was seen at 2.15 on February 25, 1999 when the crisis occurred. The drift α is taken to be the short-term interest rate differential between the BLR and the USD, and is set to be –4%. The volatility σ_S of the BLR/USD rate used is the market historical volatility, which

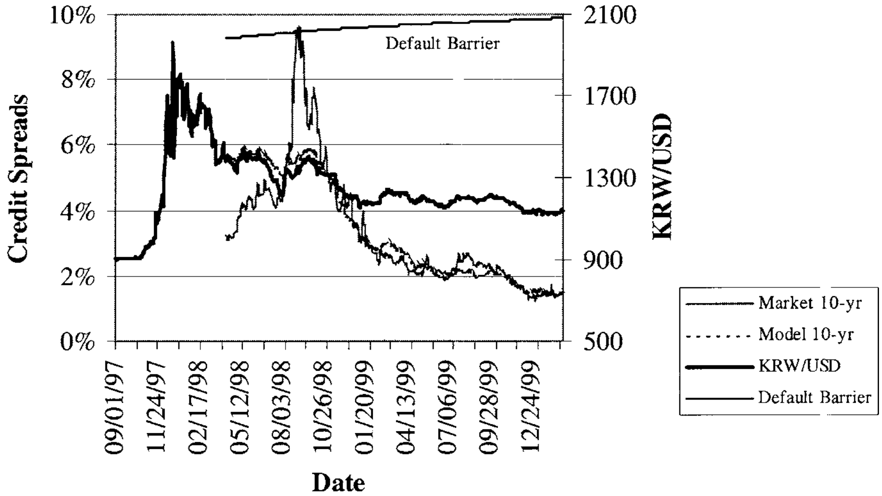

Figure 2. Market and model credit spreads of a 10-year USD denominated South Korean Government bond with $S_0 = 1982$, $\beta = -1$, $\alpha = -2\%$, $W = 0.4$, $\sigma_r = 0.078$, $r = 5\%$, $\theta = 9\%$ and $\kappa = 0.5$.

is approximated as decaying with the rate of exp (-5τ) from 25% as at January 7, 1999 to 9% as at March 3, 2000, where τ is the time from January 7, 1999. Since the BLG bonds were rated B+ when they were issued, value W is set to be 0.3. The default barrier is assumed to mainly follow the mean drift of S and β is set to be –1. Other parameters used in the calculations are the same as those used in Figures 1 and 2.

The model values and market credit spreads of the 5-year and 10-year BLG bonds are illustrated in Figures 3 and 4 respectively. The figures also include the BLR/USD exchange rates during and after the crisis period. Figure 3 shows that the model values of the 5-year BLG bond follow the market credit spreads closely from April to August 1999, but the model values are much higher than the market credit spreads afterwards until January 2000. Figure 4 shows that the model values broadly track the market credit spreads of the 10-year BLG bond, however, their differences increase from October 1999 to January 2000. The discrepancies between the market and model values, especially in Figure 3, show the limitation of just using the foreign exchange rate as the signaling variable in the model. Other financial factors might have significant impact on the credit quality of the bonds. However, the results still show the influence of the foreign exchange rate on the credit spreads, which is considered as an important signaling factor of the model.

The above numerical results illustrate that the model values can broadly follow the market bond data. It is noted that the initial S_0 of the default barrier affects the default probability of the bonds in the model. S_0 is set to be the particular values (i.e., the 1982 KRW/USD for the SKG bonds and 2.4 BLR/USD for the BLG bonds) which are slightly above the weakest exchange rates of the two currencies during the financial crisis in the two countries. The known levels of the

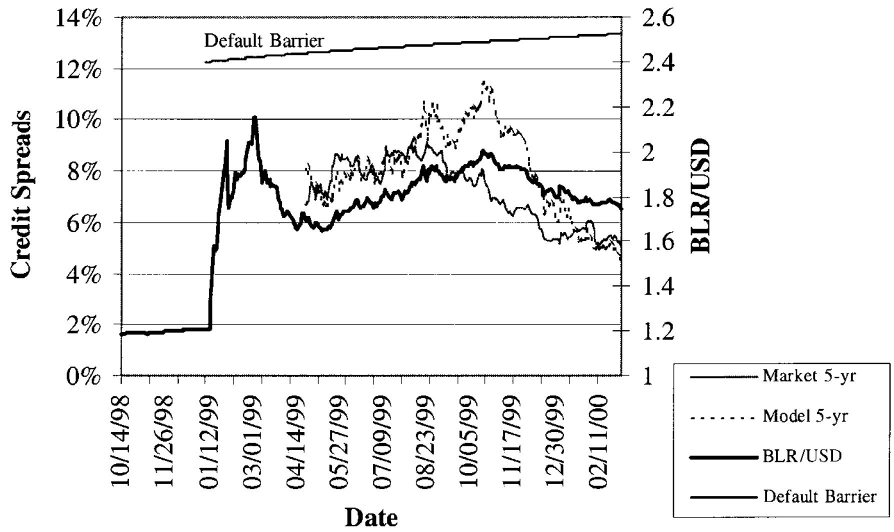

Figure 3. Market and model credit spreads of a 5-year USD denominated Brazilian Government bond with $S_0 = 2.4$, $\beta = -1$, $\alpha = -4\%$, $W = 0.3$, $\sigma_r = 0.078$, $r = 5\%$, $\theta = 9\%$ and $\kappa = 0.5$.

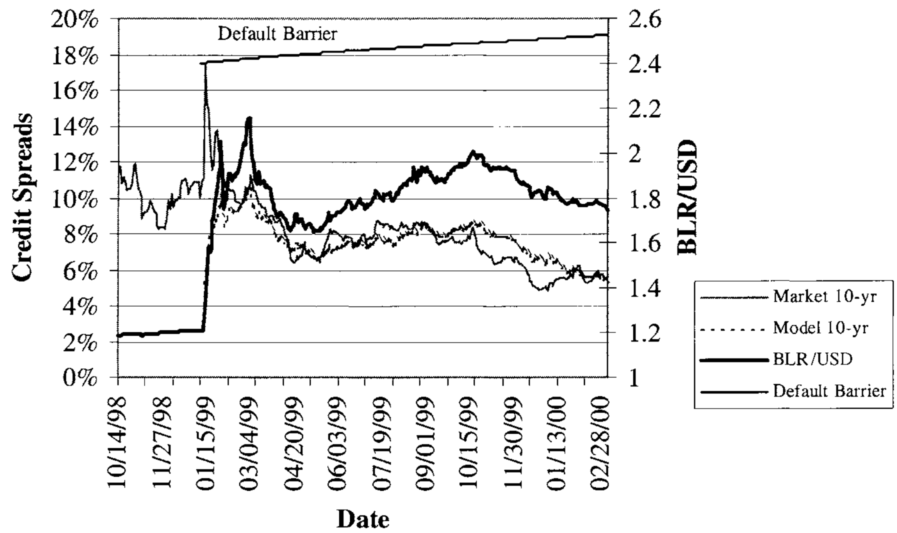

Figure 4. Market and model credit spreads of a 10-year USD denominated Brazilian Government bond with $S_0 = 2.4$, $\beta = -1$, $\alpha = -4\%$, $W = 0.3$, $\sigma_r = 0.078$, $r = 5\%$, $\theta = 9\%$ and $\kappa = 0.5$.

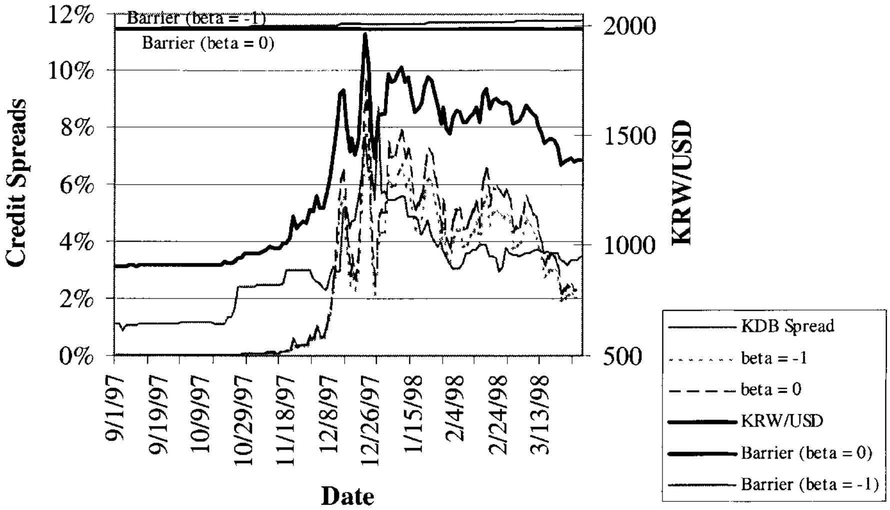

Figure 5. Market and model credit spreads of a 6-year USD denominated bond issued by Korea Development Bank with $S_0 = 1982$, $\beta = -1$ and 0, $\alpha = -2\%$, $W = 0.4$, $\sigma_r = 0.078$, $r = 5\%$, $\theta = 9\%$ and $\kappa = 0.5$.

exchange rates that would cause default or distress of the bonds make the model work well. The numerical results also demonstrate that the market prices in the foreign exchange market and the bond market in these two countries were highly correlated during and after the financial crisis; therefore, the credit spreads of the bonds can be modeled by the dynamics of the foreign exchange rates.

In order to study the behavior of credit spreads of a defaultable bond under stressed market conditions, model values of a 6-year USD bond issued by Korea Development Bank (KDB, a state-owned bank) are obtained using the KRW/USD exchange rates during the crisis period in 1997. During that period, the liquidity of the Korean bond market was very thin. The model credit spreads together with the exchange rates are illustrated in Figure 5 using $\beta = 0$ and –1 respectively. The volatility of S is set to be 10% because of the low volatility of the exchange rate before the crisis. Since the KDB bond was rated A before the crisis, value W is set to be 0.5. Other parameters used are the same as those used in Figures 1 and 2. Figure 5 shows that when the KRW/USD surges towards the default barrier, the model credit spreads increase to 8.09% and 9.60% for $\beta = -1$ and $\beta = 0$ respectively. During the crisis period from December 1997 to February 1998, the credit spreads of the KDB bond had increased to 7.65% and 8.73%. Although the model values are usually higher than the market values, in the case of $\beta = -1$ both values broadly move with quite narrow margins. The model values show that defaultable bonds under stressed market conditions can be valued by using the model in which the signaling variable is identified as some empirical market factors such as foreign exchange rates. Therefore, for a defined stress scenario where the market factors are set as certain levels, the stress-testing results of defaultable bonds can be obtained from the valuation model.

4. Summary

This paper develops a valuation model to value defaultable bonds in emerging markets. Default occurs when some signaling process hits a low default barrier. The signaling variable is considered to be some macro-economic variables such as foreign exchange rates. The dynamics of the default barrier depend on the volatility and the drift of the signaling variable. Since the volatility of the signaling variable affects the level of the default barrier over time, more realistic default scenarios can be put into the model through adjusting the barrier's dynamics. We derive a closed-form solution of the defaultable bond price from the model as a function of a signaling variable and a short-term interest rate. The model parameters are time dependent. The numerical results show that the model values generated by using foreign exchange rates as the signaling variables can broadly track the market credit spreads of defaultable bonds in emerging markets, such as South Korea and Brazil. The numerical results also show that defaultable bonds under a stressed market situation can be valued by using the model. Future research is needed to identify what other empirical market factors can be incorporated into the signaling variable for the valuation model of defaultable bonds in emerging markets.

Appendix

In our model of defaultable bonds, the bond price P, which is a function of the value S of a signaling variable determining the occurrence of default, the short-term interest rate r and the time to maturity t, is governed by the partial differential equation (see Equation (3) in the main text)

$$\begin{aligned} \frac{\partial P(S,r,t)}{\partial t} &= \frac{1}{2}\sigma_S(t)^2 S^2 \frac{\partial^2 P}{\partial S^2} + \frac{1}{2}\sigma_r(t)^2 r \frac{\partial^2 P}{\partial r^2} + \alpha(t) S \frac{\partial P}{\partial S} + \\ &\quad + \kappa(t)[\theta(t) - r]\frac{\partial P}{\partial r} - rP . \end{aligned} \tag{A.1}$$

To solve this partial differential equation, we first rewrite it in terms of the variable $x = \text{In} S$ as follows:

$$\begin{aligned} \frac{\partial P(e^x,r,t)}{\partial t} &= \frac{1}{2}\sigma_S(t)^2 \frac{\partial^2 P}{\partial x^2} + \frac{1}{2}\sigma_r(t)^2 r \frac{\partial^2 P}{\partial r^2} \\ &\quad \left[\alpha(t) - \frac{1}{2}\sigma_S(t)^2\right]\frac{\partial P}{\partial x} + \kappa(t)[\theta(t) - r]\frac{\partial P}{\partial r} - rP . \end{aligned} \tag{A.2}$$

Since the variables x and r are separable, and the boundary conditions for r are: (a) $P(e^x, r, t)$ is finite as $r \to 0$, and (b) $P(e^x, r \to \infty, t) = 0$, the price function $P(e^x, r, t)$ can be expressed as the product $Q(r,t)\,F(x,t)$, where $Q(r,t)$ is the price of a default-free bond function of the CIR model with explicitly time-dependent parameters (Cox, Ingersoll and Ross, 1985), and $F(x,t)$ satisfies the equation

$$\frac{\partial F(x,t)}{\partial t} = \frac{1}{2}\sigma_S(t)^2 \frac{\partial^2 F}{\partial x^2} + \left[\alpha(t) - \frac{1}{2}\sigma_S(t)^2\right]\frac{\partial F}{\partial x}, \tag{A.3}$$

which is simply the well-known Black–Scholes equation with time-dependent parameters in the absence of the discount term. Assuming the usual natural boundary conditions for S, the solution of Equation (A.3) is found to be (Lo and Hui, 2001; Lo, Lee and Hui, 2001)

$$F(x,t) = \int_{-\infty}^{\infty} dx' K_0\left(x,t;x',0\right) F\left(x',0\right), \tag{A.4}$$

where

$$K_0\left(x,t;x',0\right) = \frac{1}{\sqrt{4\pi B(t)}} \exp\left\{-\frac{\left[x - x' + C(t)\right]^2}{4B(t)}\right\} \tag{A.5}$$

is the kernel of the Equation (A.3), and

$$B(t) = \frac{1}{2}\int_0^t \sigma_S(\tau)^2 d\tau, \quad C(t) = \frac{1}{2}\int_0^t \left[\alpha(\tau) - \frac{\sigma_S(\tau)^2}{2}\right] d\tau. \tag{A.6}$$

Using an approach based upon the *method of images*, we can straightforwardly incorporate an absorbing moving barrier along the S-axis with drifted dynamics of the form $H(t) = S_0 \exp\left[-C(t) - \gamma B(t)\right]$ into the model, where S_0 is the pre-defined signal value of the barrier and γ is a real adjustable parameter controlling the movement of the barrier. The corresponding bond price $P(e^y, r, t)$ is then given by (Lo, Lee and Hui, 2001)

$$P\left(e^y, r, t\right) = \int_0^{\infty} dy' G\left(y,t;y',0;r\right) P\left(e^{y'}, r, 0\right)$$

$$\begin{aligned} G\left(y,t;y',0;r\right) &= Q(r,t)\left\{K_0\left(y,t;y',0\right) - K_0\left(y,t;-y',0\right)\exp\left(-\gamma y'\right)\right\} \\ &= Q(r,t)\left\{K\left(y,t;y',0\right) - K\left(y,t;-y',0\right)\right\}\exp\left[-\frac{\gamma}{2}\left(y-y'\right)\right] \times \\ &\quad \times \exp\left\{\frac{\gamma}{2}\left[C(t) + \tfrac{\gamma}{2}B(t)\right]\right\} \end{aligned}$$

$$K\left(y,t;y',0\right) = \frac{1}{\sqrt{4\pi B(t)}} \exp\left\{-\frac{\left[y - y' + C(t) + \gamma B(t)\right]^2}{4B(t)}\right\}, \tag{A.7}$$

where $y = \ln(S/S_0)$ and $y' = \ln\left(S'/S_0\right)$. It should be noted that this solution vanishes at the barrier; that is, it is the solution associated with the homogeneous boundary condition only. Nevertheless, it is an easy task to extend the solution to satisfy the inhomogeneous boundary condition: $P(S, r, t) = WFQ(r, t)$ at $S = H(t)$, by simply adding the trivial solution $WFQ(r, t)$ of the pricing equation in Equation (A.1). As a result, the bond price becomes

$$P(e^y, r, t) = WFQ(r, t) + \int_0^\infty dy' G(y, t; y', 0; r) \left[P\left(e^{y'}, r, 0\right) - WF \right]. \quad \text{(A.8)}$$

Then, by requiring that the solution associated with the inhomogeneous boundary condition to obey the prescribed final payoff condition: $P(e^y, r, t = 0) = F$, we can readily obtain the desired defaultable bond price function:

$$P\left(e^y, r, t\right) = FQ(r, t) \left\{ W + (1 - W) \left[N(\xi_1) - N(\xi_2) \exp(\xi_3) \right] \right\}, \quad \text{(A.9)}$$

where

$$\xi_1 = \frac{y + C(t)}{\sqrt{2B(t)}}, \quad \xi_2 = -\xi_1 - \gamma\sqrt{2B(t)},$$

$$\xi_3 = \gamma\left[y + C(t) + \gamma B(t))\right]. \quad \text{(A.10)}$$

and N is a cumulative normal distribution function.

For a special case of constant α and σ_S, the moving barrier will simply be reduced to an 'exponentially' moving barrier: $H(t) = S_0 \exp\left[-\beta\left(\alpha - \sigma_S^2/2\right)t\right]$, where

$$\beta = 1 + \frac{\gamma\sigma_S^2}{2\alpha - \sigma_S^2}. \quad \text{(A.11)}$$

Accordingly, the defaultable bond price is given by

$$P(S, r, t) = FQ(r, t) \left\{ W + (1 - W) \left[N(d_1) - \left(\frac{S}{S_0}\right)^{\eta_1} \exp(\eta_2) N(d_2) \right] \right\}, \quad \text{(A.12)}$$

where

$$d_1 = \frac{\ln(S/S_0) + \left(\alpha - \sigma_S^2/2\right)t}{\sigma_S\sqrt{t}}, \quad d_2 = -d_1 - 2(\beta - 1)\left(\frac{\alpha - \sigma_S^2/2}{\sigma_S}\right)\sqrt{t},$$

$$\eta_1 = -2(1 - \beta)\left(\frac{\alpha - \sigma_S^2/2}{\sigma_S^2}\right), \quad \eta_2 = \eta_1\beta\left(\alpha - \sigma_S^2/2\right)t. \quad \text{(A.13)}$$

It should be noted that when the parameter β is set equal to zero (or equivalently, $\gamma = 1 - 2\alpha/\sigma_S^2$), the case of a constant barrier is obtained, i.e., recovering the model of Cathcart and El-Jahel.[6] Obviously, our model is more general and flexible because of the presence of a risk-adjustable moving barrier, as discussed in the main text.

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Notes

1. Interest rates following the Ornstein–Uhlenbeck process are studied by Shimko et al. (1993) in valuation of corporate bonds without any default barrier.
2. See Jones et al. (1984) and Sarig and Warga (1989).
3. A default barrier with drifted dynamics governed by interest rates has been discussed in the defaultable bond pricing models proposed by Briys and de Varenne (1997) and Schöbel (1999).
4. Equation (3) is backward in time, therefore, the maturity of the bond is at $t = 0$.
5. The market data used in this paper are daily closing mid-rates based on the database of Bloomberg.
6. In fact, it can be easily shown that the inverse Laplace transform in the model of Cathcart and El-Jahel can be performed analytically, and the final result can be cast in the same closed form as ours, i.e., Equation (A.12). See, for example, T. W. Shaw, *Modelling Financial Derivatives with Mathematicsa: Mathematical Models and Benchmark Algorithms*, (Cambridge University Press, N.Y., 1998).

References

Altman, E. I. (1983/1990) *Corporate Financial Distress*, Wiley, New York.

Altman, E. I. (1991) Default and returns on high-yield bonds through the first half of 1991, *Financ. Anal. J.* **47**, 66–77.

Altman, E. I. (1992) Revisiting the high-yield bond market, *Financ. Managt* **21**, 78–92.

Artzner, P. and Delbaen, F. (1992) Credit risk and prepayment option, *ASTIN Bull.* **22**, 81–96.

Black, F. and Scholes, M. (1973) The pricing of options and corporate liability, *J. Polit. Econom.* **81**, 637–654.

Black, F. and Cox, J. (1976) Valuing corporate securities: Some effects of bond indenture provisions, *J. Finance* **35**, 1223–1234.

Briys, E. and de Varenne, F. (1997) Valuing risky fixed rate debt: An extension, *J. Financ. Quantitat. Anal.* **32**, 230–248.

Cathcart, L. and El-Jahel, L. (1998) Valuation of defaultable bonds, *J. Fixed Income* **2**, 65–78.

Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985) A theory of the term structure of interest rates, *Econometrica* **53** (2), 385–407.

Duffie, D. and Singleton, K. (1997) Modeling Term Structures of Defaultable Bonds, Graduate School of Business, Stanford University.

Jarrow, R., Lando, A., and Turnbull, S. (1994) A Markov Model for the Term Structure of Credit Spreads, Graduate School of Management, Cornell University.

Jarrow, R. and Turnbull, S. (1995) Pricing options on financial securities subject to default risk, *J. Finance* **50**, 53–86.

Jones, E. P., Mason, S. P., and Rosenfeld, E. (1984) Contingent claim analysis of corporate capital structures: An empirical investigation, *J. Finance* **39**, 611–625.

Lo, C. F. and Hui, C. H. (2001) Valuation of financial derivatives with time-dependent parameters – Lie algebraic approach, *Quantitat. Finance* **1**, 73–78.

Lo, C. F., Lee H. C., and Hui, C. H. (2001) Pricing Black–Scholes single-barrier options with time-dependent parameters, *Wilmott Magazine*, August 2001.

Longstaff, F. and Schwartz, E. (1995) A simple approach to valuing risky fixed and floating rate debt, *J. Finance* **50**, 789–819.

Madan, D. and Unal, H. (1993) Pricing the Risks of Default, College of Business, University of Maryland.

Merton, R. C. (1974) On the pricing of corporate debt: The risk structure of interest rates, *J. Finance* **2**, 449–470.

Sarig, O. and Warga, A. (1989) Some empirical estimates of the risk structure of interest rates, *J. Finance* **44**, 1351–1360.

Schöbel, F. (1999) A note on the valuation of risky corporate bonds, *OR Spektrum* **21**, 35–47.

Shimko, D. C., Tejima, N., and Van Deventer, D. R. (1993) The pricing of risky debt when interest rates are stochastic, *J. Fixed Income* **3**, 58–65.

Wilson, T. (1997a) Portfolio credit risk (1), *Risk* **10** (9), 111–116.

Wilson, T. (1997b) Portfolio credit risk (2), *Risk* **10** (10), 56–61.