Pricing Vulnerable Black-Scholes Options with Dynamic Default Barriers

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The valuation model here for European Black-Scholes options with a dynamic default barrier permits different default scenarios to be incorporated into the valuation model by adjusting the default barrier's dynamics. Closed-form solutions for vulnerable option values based on the model are derived to study the impact of the dynamic default barriers proposed in the Briys and de Varenne and Longstaff and Schwartz models. Numerical results show that the different default scenarios implied from the dynamic default barrier have a material impact on option prices.

Institutions such as banks that hold options cannot ignore the possibility of default by the option writer. Expectations of possible future losses due to default are reflected in option values. Researchers have developed models to price defaultable options, and two theoretical approaches are used to model default-triggering mechanisms.

The first approach, proposed by Merton [1974], models default that occurs at option expiration. It assumes that even if the option writer experiences financial distress, holders of options will wait until option expiration to determine their nominal claims. Using this approach, Johnson and Stulz [1987], Klein [1996], and Klein and Inglis [1999] study the impact of default risk on option prices by assuming stochastic processes for both the option writer's assets and the underlying asset value of the option. Vulnerable option prices calculated from these models are markedly different from default-free option prices based on the standard Black-Scholes [1973] model.

The second approach allows default to occur at any time prior to option maturity. Hull and White [1995] follow this approach to model default as occurring whenever the firm value of the option writer falls below a constant barrier. They assume the payoff upon default is proportional to the nominal amount of a claim. The correlation between the underlying asset and the default risk of the option writer is not considered in their closed form solutions.

Jarrow and Turnbull [1995] also consider early default in studying the effect of default risk on fixed-income and other options. They assume for simplicity that the term structure of credit spreads is independent of the term structure of risk-free interest rates.

Longstaff and Schwartz [1995] (hereafter LS) use the constant default barrier in Hull and White's formula to model risky bond prices. The constant default barrier makes the total amount of debt issued by the firm (the bond issuer) constant over time. The LS model therefore predicts that the expected leverage ratio of the firm will decline exponentially over time, although this decline is not supported by empirical observation.

Briys and de Varenne [1997] (hereafter BV) take a different approach. They propose a default barrier that grows with time together with the firm value. The expected level of
leverage is therefore kept constant. This assumption may be appropriate if the firm is either unwilling or unable to reduce its expected level of leverage over time.

Both the LS and BV models specify the time path for the default barrier, i.e., a constant value in the LS model, and a barrier rising at the risk-free interest rate in the BV model. With such simple dynamics, the default barriers in these models are not flexible enough to incorporate other default scenarios. That is, default could be triggered even when firm value is above the total amount of debt issued by the firm because of a liquidity problem (such as repayment of short-term debt). In this case, the default barrier and the default probability could be higher than defined in the LS model.

Our main objective is to develop a valuation model for European options incorporating a dynamic default barrier that is more flexible than the barrier in the LS or BV model. The movement of the default barrier in our proposed model is governed by the constant risk-free interest rate and the variance of the firm value of the option writer. It is further adjusted by a free parameter $\beta$.

When $\beta$ is set to be positive, the default barrier in our proposed model will be lower than in the BV model. This implies in turn that the default probability of an option writer is lower than that indicated by the BV model.

The dynamics of the default barrier can thus be adjusted by the parameter $\beta$ to incorporate different default scenarios. The parameter $\beta$ together with the effect of the variance of firm value can be interpreted as a measure of how much the dynamics of the default barrier deviate from the default barrier assumed in the BV model. We show in that, using an appropriate value of $\beta$, our model also encompasses the constant default barrier of the LS model.

The payoff conditions in the dynamic default barrier model follow the conditions proposed by Hull and White [1995]. When default occurs, the option holder receives an exogenously specified fraction of the default-free option value. When the option expires without default, the payoff is the regular payoff of the option. Given these payoff conditions, closed-form solutions of vulnerable European option values are derived.

Since early default is incorporated into the model, in the case of a series of options written by an option writer, default on one of the options will trigger default on the other options at the same time. The model can therefore be applied to price a vulnerable option made up a series of options, such as an interest rate cap or floor based on Black’s [1976] model, by constructing a portfolio of options (e.g., caplets) with the dynamic default barrier.

Our work can be considered an extension of Hull and White [1995] and Klein [1996].

I. MODEL OF VULNERABLE EUROPEAN OPTIONS

The model is based on the contributions of Hull and White [1995], Longstaff and Schwartz [1995], Klein [1996], and Briys and de Varenne [1997]. A continuous-time framework is used to price a European option with default risk. Let $V$ denote the value of the underlying asset of the option and $Q$ denote the firm value of the option writer. $V$ and $Q$ are assumed to follow lognormal diffusions modeled by these stochastic differential equations:

$$\frac{dV}{V} = \mu_V dt + \sigma_V dZ_V$$

$$\frac{dQ}{Q} = \mu_Q dt + \sigma_Q dZ_Q$$

where $\mu_V$ and $\mu_Q$ are the respective drift rates, and $\sigma_V$ and $\sigma_Q$ are the respective volatility values. The Wiener processes $dZ_Q$ and $dZ_V$ are correlated, with

$$dZ_Q dZ_V = \rho_{QV} dt$$

The price of the vulnerable option is $P(Q, V; t)$. Using Itô’s lemma and the standard risk-neutral arguments, the partial differential equation governing the option value is

$$\frac{\partial P}{\partial t} + \frac{1}{2} \sigma_Q^2 V^2 \frac{\partial^2 P}{\partial Q^2} + \frac{1}{2} \sigma_V^2 V^2 \frac{\partial^2 P}{\partial V^2} + \rho_{QV} \sigma_Q \sigma_V V \frac{\partial P}{\partial Q} \frac{\partial P}{\partial V} + rQ \frac{\partial P}{\partial Q} + (r-d)V \frac{\partial P}{\partial V} - rP$$

where $r$ is the risk-free interest rate, and $d$ is dividend yield. If $V$ is a stock price, the option value is obtained by solving Equation (4) subject to the final payoff condition and the boundary condition imposed by a default barrier.
**Exhibit 1**

Movements of Dynamic Default Barrier Over One Year

\[ \beta = 0.4879 \text{ (LS model), } 0.022, \text{ and } -0.00125 \text{ (BV model). All default barriers at } t = 1 \text{ are set at 90. Other input parameters are } r = d = 0.03 \text{ and } \sigma_Q = 0.05. \]

When \( Q \) breaches the default barrier \( X(t) \), default occurs before option maturity. The payoff to the option holder is then specified by

\[ P(Q = X, V, t) = \gamma C(V, t) \quad \gamma < 1 \tag{5} \]

where \( \gamma \) is the proportional recovery made in the event of default, and \( C(V, t) \) is a default-free option value.

If \( Q \) never reaches the default barrier, the payoff to the option holder at option maturity is

\[ P(Q, V, t = 0) = \max(V - K, 0) \tag{6} \]

for a call option. The payoff conditions in Equations (5) and (6) are also proposed by Hull and White [1995].

Contrary to the LS model, where the default barrier \( X_{LS} \) is a constant, in the BV model, the default barrier grows at the interest rate \( r \) as the firm value of the option writer is expected to grow under the risk-neutral measure. Given that the current level of the default barriers in the two models are identical at the initial time:

\[ X_{LS} = X_{BV} e^{-\alpha t} \tag{7} \]

in a constant interest rate environment, the default probability is higher in the BV model than in the LS model because the default barrier \( X_{BV} \) is lower than \( X_{LS} \) as option maturity approaches (i.e., \( t = 0 \)).

The dynamic default barrier is assumed to be determined by both the interest rate and the volatility of the firm value of the option writer. It is specified in the form:

\[ X(t) = Q_0 \exp \left[ \left( \beta - r + \frac{1}{2} \sigma_Q^2 \right) t \right] \tag{8} \]

where \( Q_0 \) is the level of the default barrier at option maturity \( t = 0 \), and \( \beta \) is a real number parameter to adjust the dynamics of the default barrier.

Exhibit 1 shows the movements of the dynamic default barrier over time according to Equation (8) using \( \beta \) of 0.04879, 0.022, and -0.00125. Other input parameters are \( r = d = 0.03 \) and \( \sigma_Q = 0.05 \). The default barrier at \( t = 1 \) is 90 for all the default barriers.

Exhibit 1 demonstrates that the level of the default barrier in the BV model (i.e., \( \beta = -0.00125 \)) increases with time to option maturity according to the risk-free interest rate. The level of the default barrier with \( \beta = 0.022 \) also increases with time to option maturity, but it is lower than the level in the BV model. When \( \beta \) is set equal to 0.04879, the default barrier is a constant over time as in the LS model.

The level of the default barrier defined in Equation (8) increases with negative \( \beta \), which implies a higher default probability of the option writer. Option writers
subject to different default scenarios can be represented by different \( \beta \). The level of the default barrier increases near option maturity, because there is uncertainty about the ability of the option writer to pay the obligation.

This is similar to the phenomenon of crisis-at-maturity for corporate bonds discussed by Johnson [1965]. He assumes that the bond issuer is unable to accumulate cash for debt repayment before bond maturity. In other words, an option writer with negative \( \beta \) (i.e., a high default barrier near option maturity) is susceptible to a short-term liquidity problem that could trigger a default. (Dynamic default barriers with \( \beta \) of 0.004879, 0.0022, and -0.00125 that cause different percentage reductions in option prices due to default risk are shown later.)

The solution of Equation (4) subject to the boundary condition of the dynamic default barrier of Equations (5) and (8), and the final payoff condition of Equation (6) for a call option, is

\[
P(Q, V, t) = e^{-\gamma} \sum_{n=1}^{\infty} I_n + \gamma C(V, t)
\]

where

\[
I_n = (\gamma - N)exp(N\sqrt{v/2})N\left(\frac{\ln(Q/V)}{\sigma_v \sqrt{t}} + \theta, \frac{\ln(V)}{\sigma_v \sqrt{t}} - \theta\right)
\]

\[
I_n = (\gamma - N)N\left(\frac{\ln(Q/V)}{\sigma_v \sqrt{t}} + \theta, \frac{\ln(V)}{\sigma_v \sqrt{t}} - \theta\right)
\]

\[
I_n = (\gamma - N)N\left(\frac{\ln(Q/V)}{\sigma_v \sqrt{t}} + \theta, \frac{\ln(V)}{\sigma_v \sqrt{t}} - \theta\right)
\]

\[
\theta_1 = \rho_{o\sigma}, \sigma_{\gamma}, 1
\]

\[
\theta_2 = \sigma_{\gamma}, 1
\]

\[
\theta_3 = -\rho_{o\sigma}, \sigma_{\gamma}, 1
\]

\[
\theta_4 = \sigma_{\gamma}, 1
\]

\[
\theta_5 = \beta, \sigma_{\gamma}, 1
\]

\[
\theta_6 = -\rho_{o\sigma}, \sigma_{\gamma}, 1
\]

\[
\theta_7 = \beta, \sigma_{\gamma}, 1
\]

\[
C(V, \theta) \text{ is the default-free call option value, and } N_\theta \text{ is the bivariate cumulative normal distribution function. A detailed derivation of the solution to Equation (9) is given in the appendix. The solution for a vulnerable put option with a dynamic default barrier can be obtained by using the final payoff condition of } P(Q, V, t = 0) = \max(K - V, 0) \text{ instead of Equation (6).}
\]

II. NUMERICAL RESULTS

We study the impact of default risk due to the dynamic default barrier in a numerical example based on one-year call option prices. The strike price \( K \) and current value \( V \) of the underlying asset of the option are 1.0. The annual interest rate \( r \) and dividend \( d \) are 5%. The volatility \( \sigma_v \) of the underlying asset of the option is 15% per year. The firm value \( Q \) of the option writer has an initial value of 100 and volatility \( \sigma_Q \) of 5%.

Exhibit 2 characterizes the one-year call option with different values of the correlation \( \rho_{QV} \), between \( Q \) and \( V \) and the dynamic default barrier at \( \beta = 0.04879, 0.022, \) and -0.00125. The movements of the default barriers are therefore the same as those illustrated in Exhibit 1.

The default barrier at \( t = 1 \) is set at \( X(t = 1) = 90, 92, 94, \) and 96. The option holder is assumed to receive no recovery, i.e., \( \gamma = 0 \), in the event of default. The default-free price of the call option is 0.0567. The numerical results show the percentage reductions in the option prices due to default risk.

The percentage reductions increase with declining (more negative) correlation \( \rho_{QV} \) for the three values of \( \beta \). This phenomenon is also observed by Hull and White [1995] where the default barrier is a constant value. A negative \( \rho_{QV} \) implies that the firm value \( Q \) declines toward the default barrier when the call option becomes in the money with increasing \( V \). Thus, as the default probability of the option writer increases, the amount of loss attributable to default risk also rises. Moreover, the impact of default risk on the option prices based on \( \beta = 0.022 \) and -0.00125 (BV model) is significantly higher than that based on \( \beta = 0.04879 \) (LS model), particularly with higher default barrier values at \( t = 1 \) and negative correlation \( \rho_{QV} \).

The results illustrated in Exhibit 2 are consistent with the movements of the default barrier shown in Exhibit 1, where the level of the default barrier increases with smaller values of \( \beta \). The increasing level of the default barrier over time implies that higher default risk reduces option prices more.
EXHIBIT 2
Percentage Reductions in Price of a One-Year Call Option

| \( \rho_{QV} \) | Barrier Level at \( t = 1 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
|                 | 90              | 92              | 94              | 96              |
| -0.8 LS         | 2.09%           | 7.18%           | 19.52%          | 42.20%          |
|                 | 19.38%          | 38.93%          | 62.85%          | 83.62%          |
|                 | 7.21%           | 18.78%          | 38.85%          | 64.30%          |
| 0.4 LS          | 1.01%           | 3.67%           | 10.87%          | 26.78%          |
|                 | 10.36%          | 22.75%          | 41.38%          | 63.64%          |
|                 | 3.63%           | 10.14%          | 23.20%          | 44.24%          |
| 0.0 LS          | 0.31%           | 1.36%           | 4.90%           | 15.12%          |
|                 | 4.29%           | 11.18%          | 24.28%          | 44.74%          |
|                 | 4.29%           | 11.84%          | 27.70%          |                 |
| 0.4 LS          | 0.04%           | 0.26%           | 1.45%           | 6.85%           |
|                 | 0.99%           | 3.67%           | 10.98%          | 27.06%          |
|                 | 0.21%           | 1.07%           | 4.30%           | 14.36%          |
| 0.8 LS          | 0.00%           | 0.01%           | 0.17%           | 2.07%           |
|                 | 0.03%           | 0.32%           | 2.34%           | 11.46%          |
|                 | 0.00%           | 0.06%           | 0.65%           | 4.95%           |

Default risk based on dynamic default barrier with \( \beta = 0.04879 \) (LS model), \( \beta = 0.022 \), and \( \beta = 0.01125 \) (BV model). Correlation between firm value \( Q \) of the option writer and underlying asset \( V \) is \( \rho_{QV} \). All default barriers at \( t = 1 \) are set at 90, 92, 94, and 96. Other input parameters are \( V = K = 1, Q = 100, r = d = 0.05, \sigma_Q = 0.05, \) and \( \sigma_V = 0.15 \). The option holder makes no recovery, i.e., \( \gamma = 0 \), in the event of default. The default-free price of the call option is 0.0567.

The numerical results in Exhibits 1 and 2 show that different default scenarios presented by the dynamic default barrier have different impacts on option prices due to default risk.

As the recovery values upon default are defined by an exogenous factor and are proportional to the default-free option prices, the percentage changes in the option prices will be reduced with a corresponding increase in the recovery \( \gamma \). For example, if the proportional recovery \( \gamma \) increases from 0% to 50%, the changes in option prices will be reduced by half.

III. SUMMARY

We have developed closed-form pricing formulas for vulnerable European Black–Scholes options incorporating a dynamic default barrier. Different default scenarios can be incorporated into the valuation model by adjusting the default barrier's dynamics.

The numerical results show that the different default scenarios implied from a dynamic default barrier have a material impact on option prices. The model can be used to price other vulnerable options such as interest rate caps and floors by constructing a series of options using a dynamic default barrier.

APPENDIX

DERIVATIONS

Valuation of a Distressed Option

The price \( P \) of a distressed option with stochastic liability, which is a function of the firm value \( V \) of the underlying asset, the firm value \( Q \) of the option writer, and the time to maturity \( t \), is governed by the partial differential equation:

\[
\frac{\partial P(Q, V, t)}{\partial t} = \frac{1}{2} \sigma^2_Q(t) Q^2 \frac{\partial^2 P}{\partial Q^2} + \frac{1}{2} \sigma^2_V(t) V^2 \frac{\partial^2 P}{\partial V^2} + r(t) \rho_{QV} \sigma_Q(t) \sigma_V(t) Q V \frac{\partial P}{\partial Q} \frac{\partial P}{\partial V} + \frac{|r(t) - d(t)|}{V(t)} \frac{\partial P}{\partial V} + \sigma_Q(t) \frac{\partial P}{\partial Q} - r(t) P
\]

(A-1)

In general, all the model parameters are assumed to be explicitly time-dependent.

To solve this partial differential equation, we first rewrite it in terms of the variables \( x_1 = \ln(Q/Q_0) \) and \( x_2 = \ln(V/V_0) \), where \( Q_0 \) and \( V_0 \) are constants, as follows:
\[
\frac{\partial P(x_1, x_2, t)}{\partial t} = \frac{1}{2} \sigma_Q(t) \frac{\partial^2 P}{\partial x_1^2} + \frac{1}{2} \sigma_V(t) \frac{\partial^2 P}{\partial x_2^2} + 
\rho_{QV}(t) \sigma_Q(t) \sigma_V(t) \frac{\partial P}{\partial x_1} \frac{\partial P}{\partial x_2} + 
\left[ \sigma_V(t)^2 \left( \frac{1}{2} \sigma_Q(t) \frac{\partial P}{\partial x_1} + \frac{1}{2} \sigma_Q(t) \frac{\partial P}{\partial x_2} \right) \right] + 
\left[ \sigma_V(t)^2 \left( \frac{1}{2} \sigma_V(t) \frac{\partial P}{\partial x_1} + \frac{1}{2} \sigma_V(t) \frac{\partial P}{\partial x_2} \right) \right] - r(t) P
\]

(A-2)

According to Lo and Hui [2001], the solution of Equation (A-2) can be formally given by:

\[
P(x_1, x_2, t) = \int_{-\infty}^{\infty} dx_1' \int_{-\infty}^{\infty} dx_2' G(x_1, x_2, t; x_1', x_2') P(x_1', x_2', 0)
\]

(A-3)

where

\[
G(x_1, x_2, t; x_1', x_2') = \exp \left\{ \frac{-\int_0^t dt' \left( \eta(t') \right)}{\sqrt{4\pi c_1(t)} \sqrt{4\pi c_2(t)} \sqrt{1 - \eta^2(t)}} \times \right. 
\exp \left\{ \frac{z_1(t) z_2(t)}{c_1(t) \left( 1 - \eta^2(t) \right)} \right\} \times 
\exp \left\{ \frac{z_2(t)}{4c_2(t) \left( 1 - \eta^2(t) \right)} \right\} \times 
\exp \left\{ \frac{z_1(t)}{4c_1(t) \left( 1 - \eta^2(t) \right)} \right\}
\]

(A-4)

is the kernel of the pricing equation in Equation (A-2) and:

\[
\eta(t) = \frac{c_3(t)}{2 \sqrt{c_1(t)c_2(t)}}
\]

\[
z_1(t) = x_1' - x_1 - c_4(t) z_2(t) = x_1' - x_2 - c_5(t)
\]

\[
c_1(t) = \int_0^t dt' a_1(t') c_2(t) = \int_0^t dt' a_2(t')
\]

\[
c_3(t) = \int_0^t dt' a_3(t')
\]

\[
c_4(t) = \int_0^t dt' \left[ r(t') - a_1(t') \right]
\]

\[
c_5(t) = \int_0^t dt' \left[ r(t') - d(t') - a_2(t') \right]
\]

\[
a_1(t) = \frac{1}{2} \sigma_Q(t) \sigma_V(t)
\]

\[
a_2(t) = \frac{1}{2} \sigma_V(t) \sigma_Q(t)
\]

\[
a_3(t) = \rho_{QV}(t) \sigma_Q(t) \sigma_V(t)
\]

(A-5)

### Imposing an Early Default Barrier

For the special case of constant \(\sigma_Q, \sigma_V\), and \(\rho_{QV}\), we can apply the method of images to incorporate an absorbing barrier, i.e., a default barrier, along the \(x_1\)-axis with a drifted dynamics of the form \(X_1(t) = x_{10} - z_1(t) + Bt\) into our model, where \(x_{10} = \ln(Q_0)\) is the predefined position of the barrier, and the parameter \(\beta\) is a real adjustable parameter controlling the movement of the barrier.

The corresponding option price is then given by:

\[
P_{\text{homo}}(x_1, x_2, t) = \int_{-\infty}^{\infty} dx_2' \int_{-\infty}^{\infty} dx_1' \mathcal{K}(x_1, x_2, t; x_1', x_2') \times 
P_{\text{homo}}(x_1, x_2, 0)
\]

\[
\mathcal{K}(x_1, x_2, t; x_1', x_2') = G(x_1, x_2, t; x_1', x_2') - G(x_1, x_2, t; -x_1', x_2) - \frac{a_3}{a_1} x_2' \exp \left\{ \frac{\beta x_2'}{a_1} \right\}
\]

(A-6)

where \(z_1' = \ln(Q'/Q_0)\) and \(z_2' = \ln(V'/V_0)\).

It should be noted that this solution vanishes at the barrier; that is, it is the solution associated with the homogeneous boundary condition only. Nevertheless, it is an easy task to extend the solution to satisfy the inhomogeneous boundary condition stated in Equation (5), by simply adding the trivial solution \(\gamma C(x_2, 0)\), of the pricing equation in Equation (A-2). Here \(C(x_2, 0)\) is the no-default option value and does not depend on \(x_1\), and \(\gamma\), which lies between 0 and 1, is the proportional recovery made in the event of an early default.

Following the same procedure as shown above, the \(C(x_2, t)\) can be formally expressed as:

\[
C(x_2, t) = \int_{-\infty}^{\infty} dx_2' \mathcal{G}(x_2, t; x_2') C(x_2', 0)
\]

(A-7)

where

\[
\mathcal{G}(x_2, t; x_2') = \frac{\exp \left\{ -\int_0^t dt' r(t') \frac{z_2(t')}{4\pi c_2(t)} \right\} \exp \left\{ \frac{z_2(t)}{4c_2(t)} \right\}}{\sqrt{4\pi c_2(t)}} 
\]

(A-8)

Note that \(z_2(t)\) and \(z_3(t)\) are the same as those defined above. By imposing the usual final payoff condition of a call option with a strike price \(K\):

\[
C(x_2, 0) = \begin{cases} 
0 & \text{if } V < K \\
V - K & \text{if } V \geq K
\end{cases}
\]

(A-9)
and performing the integration in Equation (A-7), we obtain the explicit expression of the no-default option value $C(x_2, t)$ as follows:

$$C(x_2, t) = \exp \left[ - \int_0^t dt' r(t') \right] \frac{1}{\sqrt{4\pi c(t)}} \int_{\infty}^{\infty} dx_2' \left\{ V_0 \exp(x_2') - K \right\} \exp \left( -\frac{x_2'^2}{4v_2(t)} \right)$$

$$= \exp \left[ - \int_0^t dt' r(t') \right] V_0 \exp \left[ x_2'' + t a_2 \right] \times$$

$$N \left( \frac{\ln \left( \frac{x_2}{V_0} \right) - x_2'' - 2t a_2}{\sqrt{2t a_2}} \right)$$

$$- \exp \left[ - \int_0^t dt' r(t') \right] K \cdot N \left( \frac{\ln \left( \frac{x_2}{V_0} \right) - x_2''}{\sqrt{2t a_2}} \right)$$

(A-10)

where $x_2'(t) = x_2 + c_2(t)$ and $N(\cdot)$ is the normalized cumulative distribution function. Accordingly, combining $P_{homo}(x_1, x_2, t)$ and $\gamma C(x_2, t)$ yields the desired price function $P(x_1, x_2, t)$ of the distressed call option, subject to the inhomogeneous boundary condition stated in Equation (5):

$$P(x_1, x_2, t) = P_{homo}(x_1, x_2, t) + \gamma C(x_2, t)$$

(A-11)

To obtain the explicit expression of the distressed option price $P(x_1, x_2, t)$, we need to impose the final payoff condition stated in Equation (6), which is equivalent to

$$P(x_1, x_2, 0) = \begin{cases} 0 & \text{if } V < K \\ V - K & \text{if } V \geq K \end{cases}$$

(A-12)

Such a final payoff condition implies that at $t = 0$, $P_{homo}(x_1, x_2, 0)$ must satisfy the prescribed condition:

$$P_{homo}(x_1, x_2, 0) = \begin{cases} 0 & \text{if } V < K \\ (1 - \gamma)(V - K) & \text{if } V \geq K \end{cases}$$

(A-13)

After performing the integration specified in Equation (A-6), $P_{homo}(x_1, x_2, 0)$ is found to be given by:

$$P_{homo}(x_1, x_2, t) = \int_0^t dx'_1 \int_{\frac{x_1}{V_0}}^{\infty} dx'_2 (1 - \gamma) V_0 \exp(x'_2) \times$$

$$G(x_1, x_2; t; x_1, x_2) - \int_0^t dx'_1 \int_{\frac{x_1}{V_0}}^{\infty} dx'_2 (1 - \gamma) K G(x_1, x_2; t; x_1', x_2') - \int_0^t dx'_1 \int_{\frac{x_1}{V_0}}^{\infty} dx'_2 (1 - \gamma) V_0 \exp(x'_2) \exp \left( -\frac{x'_2}{a_1} \right) \times$$

$$G(x_1, x_2; t; x_1, x_2) + \int_0^t dx'_1 \int_{\frac{x_1}{V_0}}^{\infty} dx'_2 (1 - \gamma) K \exp \left( -\frac{x'_2}{a_1} \right) \times$$

$$G(x_1, x_2; t; x_1, x_2)$$

(A-14)

where:

$$I_1 = (1 - \gamma) V_0 \exp \left( x_2'' + a_2 t \right) \times$$

$$N_2 \left( \frac{x_1}{\sqrt{2a_1}} - \frac{\ln \left( \frac{K}{V_0} \right) - x_2'' - \theta_2}{\sqrt{2a_2}} \right)$$

$$I_2 = (\gamma - 1) K \cdot N_2 \left( \frac{x_1}{\sqrt{2a_1}} - \frac{\ln \left( \frac{K}{V_0} \right) - x_2'' - \theta_2}{\sqrt{2a_2}} \right)$$

$$I_3 = (\gamma - 1) V_0 \exp \left( x_2'' - \frac{\beta}{a_1} x_1 + \theta_3 \right) \times$$

$$N_2 \left( \frac{\theta_3 - x_1'}{\sqrt{2a_1}} - \frac{\ln \left( \frac{K}{V_0} \right) - x_2'' - \theta_3}{\sqrt{2a_2}} \right)$$

$$I_4 = (1 - \gamma) K \exp \left( -\frac{\beta}{a_1} x_1'' + \theta_3 \right) \times$$

$$N_2 \left( \frac{\theta_3 \theta_3 - x_1''}{\sqrt{2a_1} \sqrt{2a_2}} - \frac{\ln \left( \frac{K}{V_0} \right) - x_2'' - \theta_3}{\sqrt{2a_2}} \right)$$

$$\theta_1 = a_2 t \quad \theta_2 = 2a_2 t \quad \theta_3 = (a_3 + 2\beta) t$$

$$\theta_4 = 2a_1 a_2 t - a_2 x_1''$$

$$\theta_5 = a_3 a_2 t - a_2 x_1'' + a_2 t + a_2 \beta t$$

$$\theta_6 = 2a_2 t \quad \theta_7 = a_3 \beta t - x_1''$$

$$\theta_8 = \frac{\beta}{a_1} x_1''$$

(A-15)

Here $N_2(\cdot)$ is the normalized bivariate cumulative distribution function. Note that a more explicit form of these results, namely, Equations (A-11)-(A-15), is presented in Equation (9) of the text.
ENDNOTE

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REFERENCES


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