

# Estimation of default probability by three-factor structural model

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## Abstract

This paper develops a three-factor structural model for estimating probability of default. The model incorporates the stochastic asset value of a corporate, liability and risk-free interest rate with time-dependent model parameters. A corporate defaults when its leverage ratio increases above a predefined default-triggering level. Using average market data for corporates with different external credit ratings, the three-factor model is capable of producing the term structures of probability of default for rated corporates, that are broadly matched with the average default rates of the corresponding ratings reported by Standard & Poor's. The three-factor model can be applied to the estimation of probability of default under the internal ratings-based approach proposed in the New Basel Capital Accord.

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# The conclusions herein do not represent the views of the Hong Kong Monetary Authority.

## I. INTRODUCTION

Under the proposed internal ratings-based (IRB) approach in the New Basel Capital Accord (Basel, 2001), the Basel Committee on Banking Supervision allows banks to calculate regulatory capital charges for credit risk based on banks' internal credit risk ratings for their exposures. Probability of default (PD) associated with individual grades in a bank's rating system is one of the key components in the IRB approach. The other two basic risk components are the facility's loss given default (LGD) based on such characteristics as the presence and type of collateral or other risk mitigants, and the exposure at default (EAD) which measures the

bank's exposure at the time of default. These three risk components will be converted into risk weights to be used by banks for calculating risk-weighted exposures. The amount of risk-weighted exposures determines the required regulatory capital.

Banks adopting the IRB approach should have a rating system that provides for a separate assessment of borrower and transaction characteristics. In other words, the system should distinguish between the risk of borrower default (as measured by PD) and the transaction characteristics that affect the loss severity in the event of default (as measured by LGD). This two-dimensional approach will provide assurance that the assignment of borrower ratings is not influenced by consideration of the specific structure of the transaction. The IRB banks are responsible for determining PD and demonstrating the appropriateness of techniques used for measuring PD to banking supervisors.

In the literature on credit risk models, several approaches for modeling PDs are suggested. These approaches use different assumptions and information about a firm. The structural model proposed by Merton (1974) assumes that the market value of total assets is observable in principle. Furthermore the capital structure is explicitly considered and default happens if the total asset value is lower than the value of liabilities. In contrast, reduced form models and other spread-related approaches derive relations between the credit spread of the credit risky bond and the PD (e.g., Jarrow and Turnbull (1995) and Duffie and Singleton (1997)). An essential part of the theory of these models is the risk neutral valuation under the absence of arbitrage opportunities.

This paper presents a three-factor structural model that incorporates the stochastic asset value of a firm, liability and interest rate with time-dependent model parameters. The firm's liability is governed by a mean-

reverting stochastic process. The correlation among the three stochastic variables is incorporated into the model. The firm defaults when its liability-to-asset ratio (leverage ratio) increases above a predefined default-triggering level. This is consistent with the event of bankruptcy being associated with abnormally high levels of debt relative to the market value of the firm's assets.

Empirical findings document that companies tend to gradually adjust their capital structure toward a target level of leverage (Marsh (1982), Jalilvand and Harris (1984), Auerbach (1995) and Opler and Titman (1995)). This means that a firm adjusts its outstanding debts in response to changes in its firm value in order to achieve a target level of leverage. These findings call for a stationary-leverage-ratio model for pricing corporate bonds, which has been studied by Collin-Dufresne and Goldstein (2001). Their model considers a mean-reverting drifting (associated with interest rate dynamics) liability and constant model parameters. The three-factor model presented here is, however, more general than the stationary-leverage-ratio model in terms of the dynamics of the liability and time-dependent model parameters.

A simple and easy-to-use method is provided for computing accurate PD estimates (in closed form) based on the three-factor model. The numerical results show that the estimation of PD is sensitive to the time-dependent target leverage ratio. Using average market data for corporates with different external credit ratings, the three-factor model is capable of producing the term structures of PD for rated corporates, that are broadly matched with the average default rates of different credit ratings reported by Standard & Poor's (S&P's)(2002). The three-factor model can be applied to the estimation of PD under the IRB approach proposed in the New Basel Capital Accord.

The scheme of this paper is as follows. In the following section we discuss the three-factor model and PD estimated from the model. Numerical results of PD cal-

culated from the model based on average market data are then compared with the average default rates reported by S&P's. At the end we will summarise our investigation.

## II. MODEL

A continuous-time framework is used to value PD of a firm in the model. The firm value  $V$  is assumed to follow a lognormal diffusion process. The firm liability  $Q$  is governed by a mean-reverting lognormal diffusion process. The dynamics of the risk-free interest rate  $r$  is drawn from the term structure model of Vasicek (1977), i.e. the Ornstein-Uhlenbeck process\*. Their continuous stochastic movements are modelled by the following stochastic differential equations:

$$\begin{aligned} \frac{dV}{V} &= \mu_V(t)dt + \sigma_V(t)dZ_V \\ \frac{dQ}{Q} &= [\mu_Q(t) + \kappa_Q(t) (\ln V - \ln Q)] dt + \\ &\quad \sigma_Q(t) dZ_Q \\ dr &= \kappa_r(t) [\theta_r(t) - r] dt + \sigma_r(t)dZ_r \end{aligned} \quad (2.1)$$

where  $\sigma_Q(t)$  and  $\sigma_V(t)$  are the respective volatility values,  $\mu_Q(t)$  and  $\mu_V(t)$  are the respective drift rates, and the firm liability  $Q$  is mean-reverting at speed  $\kappa_Q(t)$ . The interest rate uncertainty is driven by a Vasicek representation with the instantaneous volatility  $\sigma_r(t)$ . The short-term interest rate  $r$  is mean-reverting to long-run mean  $\theta_r(t)$  at speed  $\kappa_r(t)$ . All model parameters are explicitly time dependent. The Wiener processes  $dZ_Q$ ,  $dZ_V$  and  $dZ_r$  are correlated with

$$\begin{aligned} dZ_V dZ_Q &= \rho_{VQ}(t)dt \\ dZ_V dZ_r &= \rho_{Vr}(t)dt \\ dZ_Q dZ_r &= \rho_{Qr}(t)dt \end{aligned} .$$

We define  $R \equiv Q/V$  to be the leverage ratio and apply the Ito's lemma to derive the partial differential equation governing a corporate discount bond  $P(R, r, t)$  of the three-factor model as follows:

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\*Although this assumed process is consistent with many of the observed properties of interest rates, it can allow negative interest rates. However, this assumption may still be justifiable in the context of the valuation because given that the current value of interest rate and the mean-level are both positive, the dynamics always imply positive expected future interest rate.

$$\begin{aligned}
& \frac{\partial P(R, r, t)}{\partial t} \\
&= \frac{1}{2} \sigma_R^2(t) R^2 \frac{\partial^2 P}{\partial R^2} + \frac{1}{2} \sigma_r^2(t) \frac{\partial^2 P}{\partial r^2} + \\
& \rho_{Rr}(t) \sigma_R(t) \sigma_r(t) R \frac{\partial^2 P}{\partial R \partial r} + \\
& \kappa_Q(t) [\ln \theta_R(t) - \ln R] R \frac{\partial P}{\partial R} + \\
& \kappa_r(t) [\theta_r(t) - r] \frac{\partial P}{\partial r} - rP \quad (2.2)
\end{aligned}$$

where  $t$  is the time-to-maturity,

$$\begin{aligned}
& \sigma_R(t) \\
&= \sqrt{\sigma_Q^2(t) - 2\rho_{QV}(t)\sigma_Q(t)\sigma_V(t) + \sigma_V^2(t)} \quad , \\
& \rho_{Rr}(t) \\
&= \frac{\rho_{Qr}(t)\sigma_Q(t) - \rho_{Vr}(t)\sigma_V(t)}{\sigma_R(t)} \quad , \quad (2.3)
\end{aligned}$$

and  $\theta_R(t)$  is the target leverage ratio.

When the firm's leverage ratio is above a predefined level, bankruptcy occurs before maturity. In order to obtain a closed-form solution of the estimation of PD from the model, the predefined level  $L(t)$  of the leverage ratio upon default is specified in the form:

$$L(t) = \exp[-c_2(t) - 4\beta c_1(t)] \quad (2.4)$$

where  $\beta$  is a real parameter to adjust the movement of  $L(t)$  over time, and  $c_1(t)$  and  $c_2(t)$  are defined as:

$$\begin{aligned}
c_1(t) &= \frac{1}{2} \int_0^t d\xi \sigma_R^2(\xi) \exp[2\alpha(\xi)] \\
c_2(t) &= \int_0^t d\xi F(\xi) \exp[\alpha(\xi)] \quad (2.5)
\end{aligned}$$

with

$$\begin{aligned}
\alpha(t) &= - \int_0^t d\xi \kappa_Q(\xi) \\
F(t) &= \kappa_Q(t) \ln \theta_R(t) - \frac{1}{2} \sigma_R^2(t) + \\
& \rho_{Rr}(t) \sigma_R(t) \sigma_r(t) c_4(t) \exp[c_3(t)] \\
c_3(t) &= - \int_0^t d\xi \kappa_r(\xi) \\
c_4(t) &= - \int_0^t d\xi \exp[-c_3(\xi)] \quad . \quad (2.6)
\end{aligned}$$

The parameter  $\beta$  is adjusted such that  $L(t)$  is approximately equal to unity over time. In view of this, Eq.(2.4) does not imply any additional specification to the model. Accordingly, the corresponding PD based on Eq.(2.2) is given by

$$\begin{aligned}
& P_{def}(x, t) \\
&= 1 - \int_{-\infty}^0 dy \{G(x \exp[\alpha(t)], t; y, 0) - \\
& G(x \exp[\alpha(t)], t; -y, 0) \cdot \exp(-4\beta y)\} \\
&= 1 - N\left(-\frac{1}{\sqrt{2c_1(t)}} [x \exp(\alpha(t)) + c_2(t)]\right) + \\
& N\left(\frac{1}{\sqrt{2c_1(t)}} [x \exp(\alpha(t)) + c_2(t) + 8\beta c_1(t)]\right) \times \\
& \exp\{4\beta [x \exp(\alpha(t)) + c_2(t)] + 16\beta^2 c_1(t)\} \quad , \quad (2.7)
\end{aligned}$$

where  $x = \ln(R)$ ,  $N(\cdot)$  is the normalized cumulative distribution function, and

$$\begin{aligned}
& G(x \exp[\alpha(t)], t; y, 0) \\
&= \frac{1}{\sqrt{4\pi c_1(t)}} \times \\
& \exp\left\{-\frac{[x \exp[\alpha(t)] - y + c_2(t)]^2}{4c_1(t)}\right\} \quad . \quad (2.8)
\end{aligned}$$

A simple and easy-to-use method that has been developed by Lo et al.(2002) for solving barrier option values with time-dependent model parameters is provided for computing accurate PD estimates based on Eq.(2.7).

### III. MODEL PDS

The computed PDs for corporates with non-investment ratings (i.e. BB, B and CCC) and investment ratings (i.e. AAA, AA, A and BBB) within a period of 15 years are presented in Figure 1 and Table 1 respectively. The model PDs are compared with the cumulative average default rates of the corresponding ratings based on 9,769 companies' assigned long-term ratings from 1981 to 2001 reported by S&P's (2002). The model parameters used for individual ratings are shown in Table 2. Other common parameters used in calculations are  $\sigma_Q = 0.1$ ,  $\sigma_r = 0.03162$ ,  $\kappa_r = 1.0$ ,  $\rho_{VQ} = 0.0$ ,  $\rho_{Vr} = 0.0$  and  $\rho_{Rr} = 0.0$ . The time-dependent target leverage ratio  $\theta_R(t)$ , which is assumed to be parametrised in the following form:

$$\theta_R(t) = \theta_{R0} [1 + \eta \exp(-\gamma t)] \quad (3.1)$$

where  $\theta_{R0}$ ,  $\eta$  and  $\gamma$  are constants determined by fitting the numerical estimates with the average default rates reported by S&P's. In our search for optimal values of the three parameters, we have observed that:

1. in the first year  $\theta_R \approx 0.732$  (*i.e.* the average leverage ratio of a corporate of ‘CCC’ rating), and
2. in the fifteenth year  $\theta_R \approx 0.315$  (*i.e.* the average leverage ratio of a corporate of ‘BBB’ rating).

These two conditions will enable us to express  $\theta_{R0}$  and  $\eta$  as functions of  $\gamma$ . In other words, we are left with the parameter  $\gamma$  only. In Figure 2 the time-dependent target leverage ratios for different rated corporates are plotted. The use of  $\theta_R = 0.315$  in the fifteenth year is close to the long-term target leverage ratio observed empirically by Collin-Dufresne and Goldstein (2001). The use of high  $\theta_R$  of 0.732 in the first year is explained by the observation that default occurring at short term is mainly triggered by a corporate’s short-term liabilities. The default (or distressed restructuring of debts) occurring in the corporate is due to its liquidity problem. The observation of the liquidity problem is similar to the phenomenon of high default rates of bonds at short maturities, that is called “crisis-at-maturity” by Johnson (1965). This explanation assumes, as Johnson points out, that corporates are unable to accumulate cash for debt repayment before maturity. The potential liquidity problem causes a corporate (even with a low current leverage ratio) to have a high initial short-term target leverage ratio and thus increases the PD of the corporate at short term. The material one- to three-year default rates reported by S&P’s reflect the liquidity problems faced by those defaulted corporates in S&P’s data pool.

The depicted term structures of PD in Figure 1 exhibit steep upward slopes at short time and is gently upward-sloping for time longer than a few years. The shapes and values of the model term structures of PD are consistent with S&P’s default rates of non-investment grade corporates. Both the model and empirical values show that low quality corporates have very high default risk at short maturities. The flat-slope at longer time reflects that PD of corporates with low credit ratings will not increase significantly over time as the corporates survive at the short term.

In Table 1, the model PDs of investment grade corporates at short term are lower than but still comparable to S&P’s default rates. Both the model PDs and S&P’s default rates increase with time. Intuitively, this is because the probability that the corporate’s leverage ratio

reaches the default boundary increases over time. The percentage differences between the model and empirical values of PD for investment grade corporates (in particular for AAA/AA-rated corporates at short time) are larger than those for non-investment grade corporates. The reason is that the observations of default events occurred in investment grade corporates are rare, in particular at short time (e.g. only 14 defaults occurred in companies with original rating of AA in S&P’s data and the average time to default is 11.9 years). A very few default events occurred in AA-rated corporates may cause significant changes in the observed default rates. On the other hand, there are 574 defaults occurred in companies with original rating of B in S&P’s data and the average time to default is only 3.8 years.

#### IV. CONCLUSION

The development of the three-factor structural model is consistent with the recent empirical observations on companies’ capital structure that tends gradually towards a target level of leverage over time. The numerical results show that the estimation of PD is sensitive to the time-dependent target leverage ratio  $\theta_R(t)$ . The PD estimated from the model based upon average market data for corporates with different external credit ratings are broadly matched with the average default rates of the corresponding ratings reported by S&P’s. The three-factor model is able to estimate the PD for corporates with observed leverage ratios and can therefore be applied to the estimation of PD under the IRB approach proposed in the New Basel Capital Accord.

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Figure 1: Term structures of PD(%) for corporates with non-investment ratings

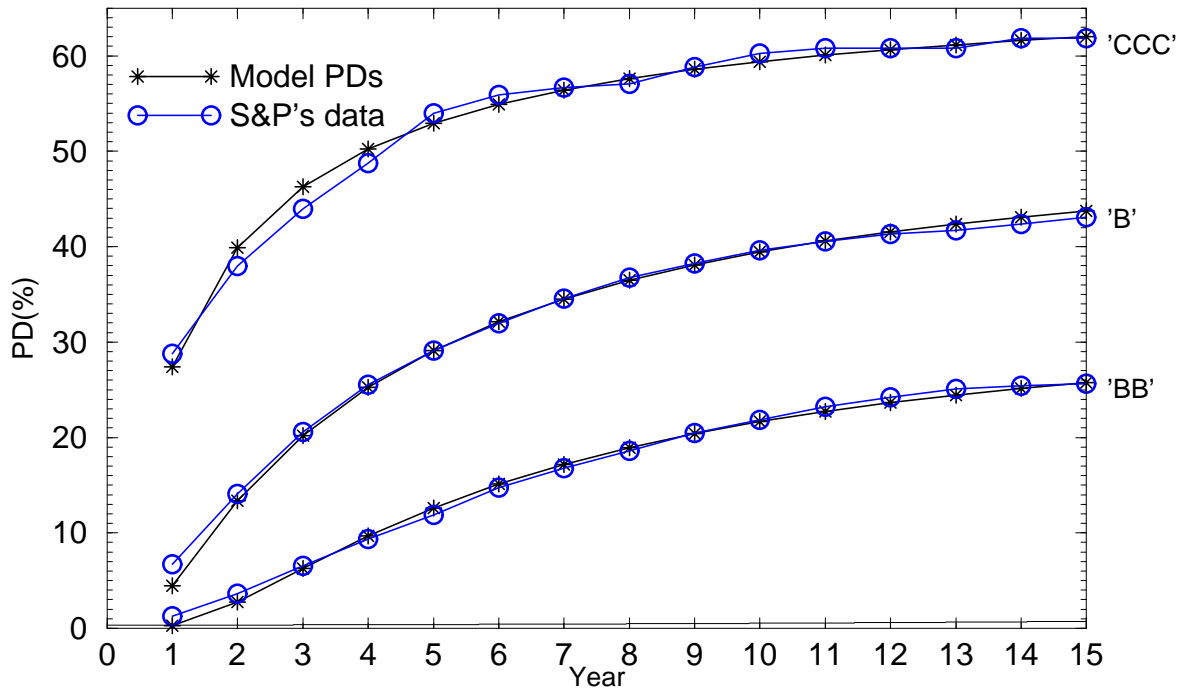
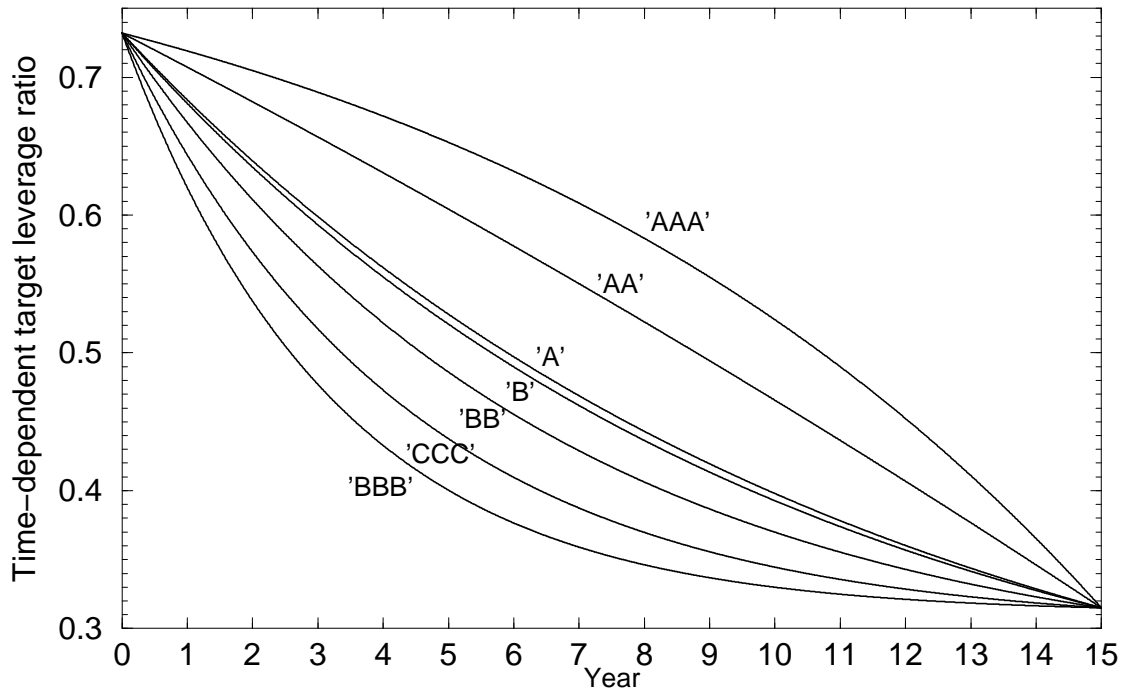


Figure 2: Term structures of time-dependent target leverage ratios for different rated corporates



Rating	Yr.1	Yr.2	Yr.3	Yr.4	Yr.5	Yr.6	Yr.7	Yr.8	Yr.9	Yr.10	Yr.11	Yr.12	Yr.13	Yr.14	Yr.15
S&P's	0.00	0.00	0.03	0.07	0.11	0.21	0.31	0.49	0.55	0.63	0.63	0.63	0.63	0.63	0.63
Model	0.0000	0.0000	0.0000	0.0000	0.0016	0.0214	0.0969	0.2387	0.4063	0.5476	0.6374	0.6806	0.6959	0.6996	0.7001
S&P's	0.01	0.03	0.09	0.16	0.26	0.39	0.56	0.71	0.82	0.99	1.14	1.32	1.43	1.56	1.72
Model	0.0000	0.0000	0.0000	0.0020	0.0255	0.1201	0.3172	0.5890	0.8735	1.1174	1.2961	1.4098	1.4729	1.5030	1.5152
S&P's	0.05	0.14	0.26	0.43	0.64	0.85	1.08	1.33	1.62	1.90	2.11	2.30	2.49	2.66	2.98
Model	0.0000	0.0001	0.0075	0.0629	0.2208	0.4939	0.8483	1.2327	1.6034	1.9326	2.2081	2.4286	2.5990	2.7271	2.8214
S&P's	0.27	0.62	0.99	1.63	2.26	3.00	3.67	4.27	4.76	5.35	6.03	6.53	7.06	7.71	8.37
Model	0.0012	0.1051	0.5298	1.2263	2.0441	2.8753	3.6650	4.3918	5.0519	5.6492	6.1904	6.6831	7.1343	7.5502	7.9360

Table 1: Model PDs (%) and default rates (%) reported by S&P's of corporates with investment ratings.

	AAA	AA	A	BBB	BB	B	CCC
Leverage ratio $R$ (%)	3.1	9.5	17.2	31.5	49.5	53.8	73.2
Volatility of the firm value $\sigma_V$	0.127	0.156	0.184	0.213	0.241	0.270	0.299
Speed of mean-reverting $\kappa_Q$	0.4	0.3	0.2	0.1	0.1	0.1	0.1
Parameter $\gamma$	0.097	0.0165	-0.09	-0.31	-0.15	-0.1	-0.23

Table 2: Parameters used in the calculations for individual ratings. The leverage ratios  $R$  are based on the industry medians reported by S&P's (2001). The values of  $\sigma_V$  fall close to the estimates of Delianedis and Geske (1999).

The values of  $\kappa_Q$  are based on the estimates of Fama and French (1999), ( $\kappa_Q \approx 0.1$ ), who investigate the universe of firms and Shyam-Sunder and Myers (1999), ( $\kappa_Q \approx 0.4$ ), whose sample includes large and financially conservative firms. (Note:  $\sigma_R = \sqrt{\sigma_V^2 + \sigma_Q^2 - 2\rho_{VQ}\sigma_Q\sigma_V}$ )